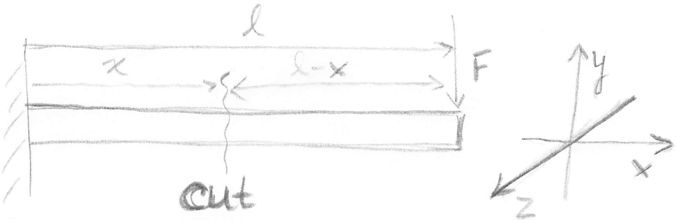
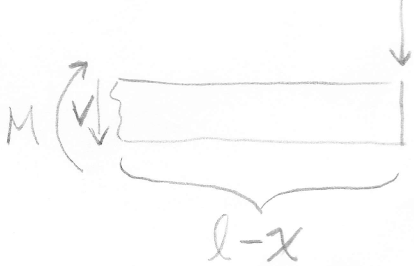
Derivations of Lab 3 Background - MAE 315

Elementary Beam theory



→ Derive the moment in the beam



↻+

$$-M - F(l-x) = 0$$

$$M = -F(l-x)$$

Bending

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI(x)}$$

No taper - I is constant along beam

* Drop negative sign in $M(x)$ w/ the understanding the deflection (y) is downward.

$$\frac{d^2y}{dx^2} = \frac{+F(l-x)}{EI}$$

Integrate

$$\frac{dy}{dx} = \frac{F}{EI}\left(lx - \frac{x^2}{2}\right) + \cancel{C_1}$$

0 slope of beam at the wall is zero!

Integrate Again

$$\frac{dy}{dx} = \frac{F}{EI}\left(lx - \frac{x^2}{2}\right)$$

$$y = \frac{F}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + \cancel{C_2}$$

∘ Deflection at the wall is zero!

Deflection at the end of the beam $\{y(x=l) = \delta\}$

$$\boxed{\delta = \frac{Fl^3}{3EI}} \rightsquigarrow \delta\left(\frac{3EI}{l^3}\right) = F \rightsquigarrow \boxed{K_{eq} = \frac{3EI}{l^3}} \ *$$

"Equivalent Spring Stiffness"

Therefore,

$$\boxed{y(x) = \delta\left(\frac{3x^2}{2l^2} - \frac{x^3}{2l^3}\right)}$$

"Deflection along the beam"

→ Diff. w/r to time

$$\dot{y}(x) = \dot{\delta}\left(\frac{3x^2}{2l^2} - \frac{x^3}{2l^3}\right)$$

$\dot{\delta}$ – Velocity of the end of the beam.

Find the Eqivalent Mass

→ Conserve Kinetic Energy

$$\cancel{\tfrac{1}{2}} M_{eq} \cancel{\dot{\delta}^2} = \rho A \int_0^l \cancel{\tfrac{1}{2}} \cancel{(\dot{\delta})^2} \left(\frac{3x^2}{2l^2} - \frac{x^3}{2l^3}\right)^2 dx$$

- ρ – Density of the beam
- A – cross sectional Area

$$m_{eq} = \rho A \int_0^l \left(\frac{9}{4}\frac{x^4}{l^4} - \frac{3x^5}{2l^5} + \frac{x^6}{4l^6} \right) dx$$

$$m_{eq} = \rho A \left(\frac{9}{20}\frac{x^5}{l^4} - \frac{3x^6}{12l^5} + \frac{x^7}{28l^6} \right) \Big|_0^l$$

$$m_{eq} = \rho A l \left(\frac{9}{20} - \frac{1}{4} + \frac{1}{28} \right)$$

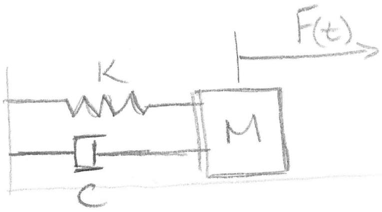
$$m_{eq} = m_{beam} \left(\frac{1}{4} \right) \left(\frac{9}{5} + \frac{1}{7} - 1 \right)$$

$$m_{eq} = m_{beam} \left(\frac{1}{4} \right) \left(\frac{63 + 5 - 35}{35} \right)$$

$$\boxed{m_{eq} = \frac{33}{140} m_{beam}} \quad *$$

"Equivalent Mass of the beam"

Solving Differential Equations Lab 3 - MAE 315



$$\boxed{\begin{array}{l}\text{Governing Equ.}\\ m\ddot{x} + c\dot{x} + Kx = F(t)\end{array}}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}F(t) \qquad \boxed{\begin{array}{l} 2\zeta\omega_n = \frac{c}{m} \\ \omega_n^2 = \frac{K}{m}\end{array}}$$

→ Sinusoidal Forcing $F(t) = F_0 \sin(\omega t)$

Suppose $x = X\sin(\omega t - \delta) = \underbrace{X\cos\delta}_{A} \sin\omega t \underbrace{- X\sin\delta}_{B} \cos\omega t$

$$\dot{x} = A\omega\cos\omega t - B\omega\sin\omega t$$
$$\ddot{x} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$$

→ Substitute:

$$-A\omega^2\sin\omega t - B\omega^2\cos\omega t + 2\zeta\omega_n(A\omega\cos\omega t - B\omega\sin\omega t) + \omega_n^2(A\sin\omega t + B\cos\omega t) = \frac{F_0}{m}\sin\omega t$$

→ Equate Coefficients

$$-A\omega^2 - 2\zeta\omega_n\omega B + \omega_n^2 A = \frac{F_0}{m}$$
$$-B\omega^2 + 2\zeta\omega_n\omega A + \omega_n^2 B = 0$$

Solve for A, B

$$B(\omega_n^2-\omega^2) = -2\zeta\omega_n\omega A$$

$$B = \frac{-2\zeta\omega_n\omega A}{\omega_n^2-\omega^2}$$

$$A(\omega_n^2-\omega^2) - 2\zeta\omega_n\omega B = \frac{F_0}{m}$$

$$A(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2 A = \frac{F_0}{m}(\omega_n^2-\omega^2)$$

$$\left.\begin{aligned} A &= \frac{F_0/m\,(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2} \\ B &= \frac{-2\zeta\omega_n\omega\left(\frac{F_0}{m}\right)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2} \end{aligned}\right\} \begin{aligned} A &= X\cos\delta \\ -B &= X\sin\delta \end{aligned}$$

$$X = \sqrt{A^2+B^2} = \frac{F_0}{m}\,\frac{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}$$

$$\boxed{X = \frac{F_0/m}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}}$$

$$\boxed{\tan\delta = -\frac{B}{A} = \frac{2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)}}$$

Laplace Solution

$F(t) = F_0 \delta(t)$
$\dot{x}(0) = 0$
$x(0) = 0$

$$\mathcal{L}(\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x)$$
$$\downarrow$$
$$s^2 X + 2\zeta\omega_n s X + \omega_n^2 X = \mathcal{L}\left(\frac{1}{m} F_0 \delta(t)\right) = \frac{F_0}{m}$$

$$X = \frac{F_0/m}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

→ Complete the squares: $s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2$

$$X = \frac{F_0/m}{(s+\zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2} \cdot \frac{\sqrt{1-\zeta^2}\,\omega_n}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$X = \frac{F_0/m}{\sqrt{1-\zeta^2}\,\omega_n} \underbrace{\frac{\sqrt{1-\zeta^2}\,\omega_n}{(s - -\zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2}}_{\frac{b}{(s-a)^2 + b^2}}$$

!!! $b = \sqrt{1-\zeta^2}\,\omega_n$
$a = -\zeta\omega_n$

$$\boxed{x(t) = \frac{F_0/m}{\sqrt{1-\zeta^2}\,\omega_n} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)}$$