Underdetermined and Overdetermined Linear Algebraic Systems

ES100

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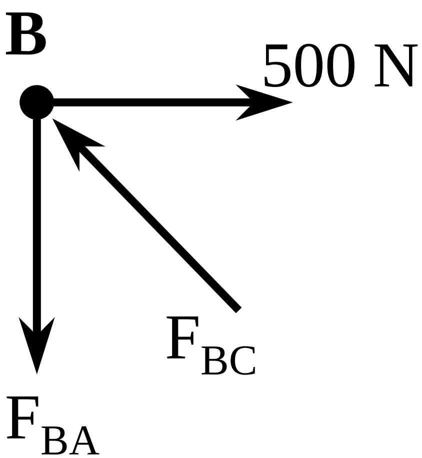
T.S. Whitten

Objectives

- Define underdetermined systems
- Define overdetermined systems
- Least Squares Examples

Review



$$\sum F_x = 0; \quad 500 - F_{BC}\sin 45° = 0$$

$$\sum F_y = 0; \quad F_{BC}\cos 45° - F_{BA} = 0$$

$$\underbrace{\begin{bmatrix} -\sin 45° & 0 \\ \cos 45° & 1 \end{bmatrix}}_{\textit{coefficients}} \cdot \underbrace{\begin{bmatrix} F_{BC} \\ F_{BA} \end{bmatrix}}_{\textit{variables}} = \begin{bmatrix} -500 \\ 0 \end{bmatrix}$$

Review cont.

$$\underbrace{\begin{bmatrix} -\sin 45^\circ & 0 \\ \cos 45^\circ & 1 \end{bmatrix}}_{coefficients} \cdot \underbrace{\begin{bmatrix} F_{BC} \\ F_{BA} \end{bmatrix}}_{variables} = \begin{bmatrix} -500 \\ 0 \end{bmatrix}$$

The system of matrices above is of the form:

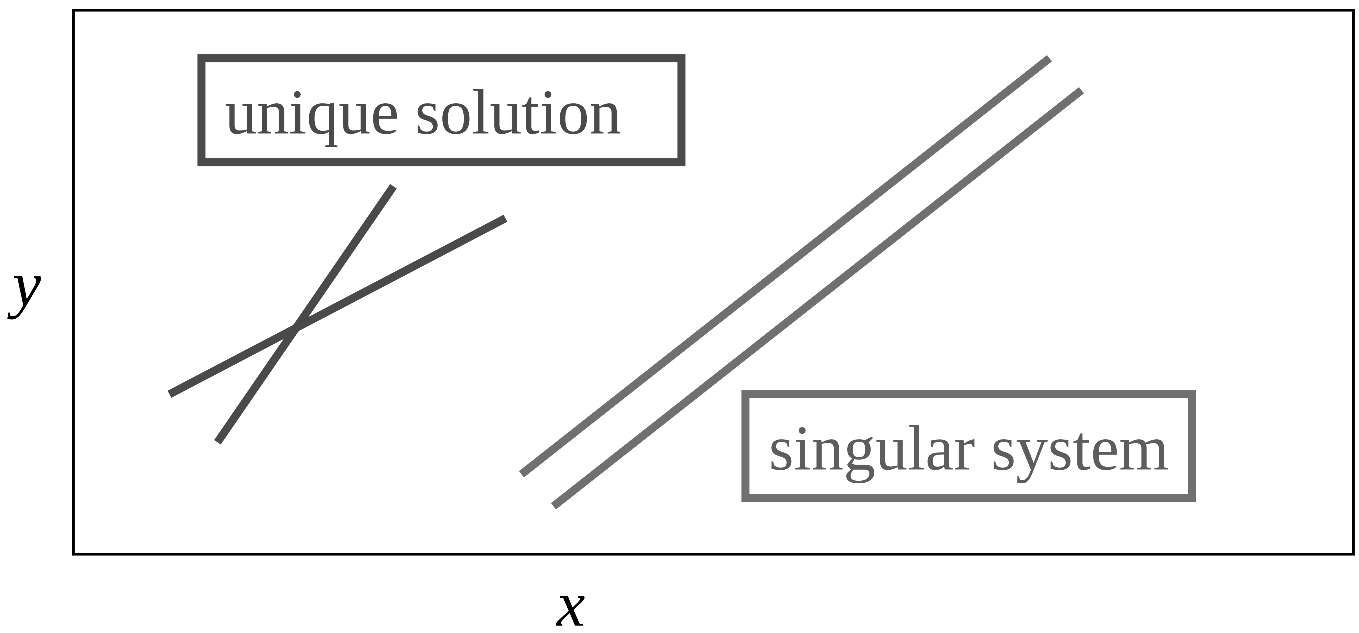
$$\mathbf{Ax} = \mathbf{b}$$

and can be solved using MATLAB *left division* thus, $x = A\backslash b$ results in a 1×2 matrix of values for F_{BC} and F_{BA}

Review Summary

- A system of two Equations and two unknowns may yield a unique solution.
  - The exception is when the determinant of ***A*** is equal to zero. Then the system is said to be singular.
- The left division operator will solve the linear system in one step by combining *two* matrix operations
  - A\B is equivalent to A^{-1}*B

Graphical Representation of Unique vs. Singular Systems



Underdetermined Systems

- A system of linear equations is may be *undetermined* if;
  1 The determinant of ***A*** is equal to zero

$$|A| = 0$$

  2 The matrix ***A*** is not square, i.e. the are more unknowns than there are equations

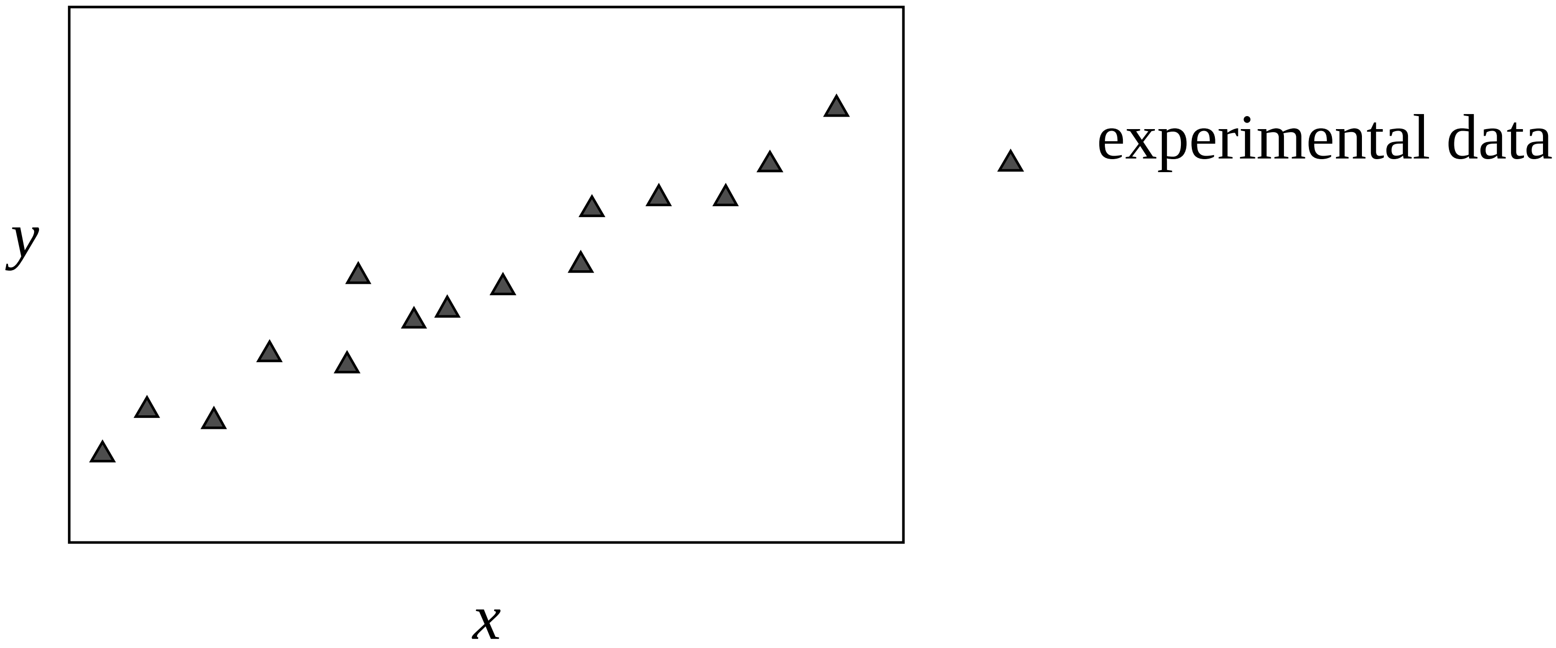
$$x + 3y + 2z = 2$$
$$x + y + z = 4$$

$$\Longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

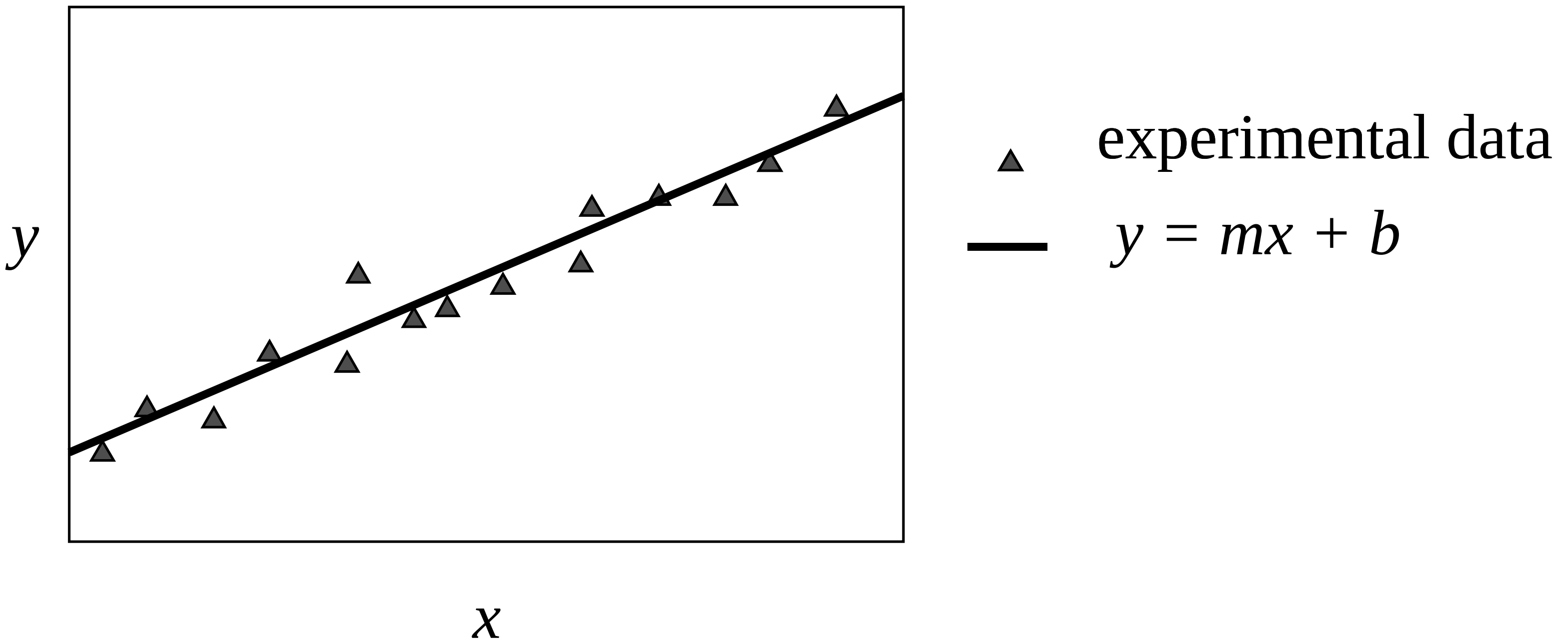
Overdetermined Systems

- The converse of an underdetermined system is an *overdetermined* system where there are more equations than there are variables
- This situation arises frequently in engineering. For example: suppose a linear relationship is expected between x and y and there are multiple data points.

Data Distribution of Linear Phenomena

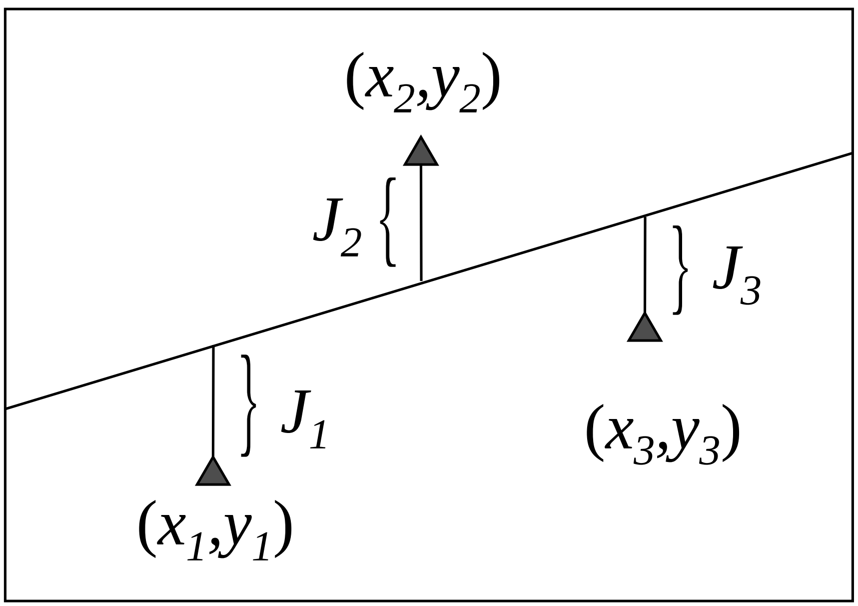


Data Distribution of Linear Phenomena



The line $y = mx + b$, that best describes this data is obtained by the *method of least squares*

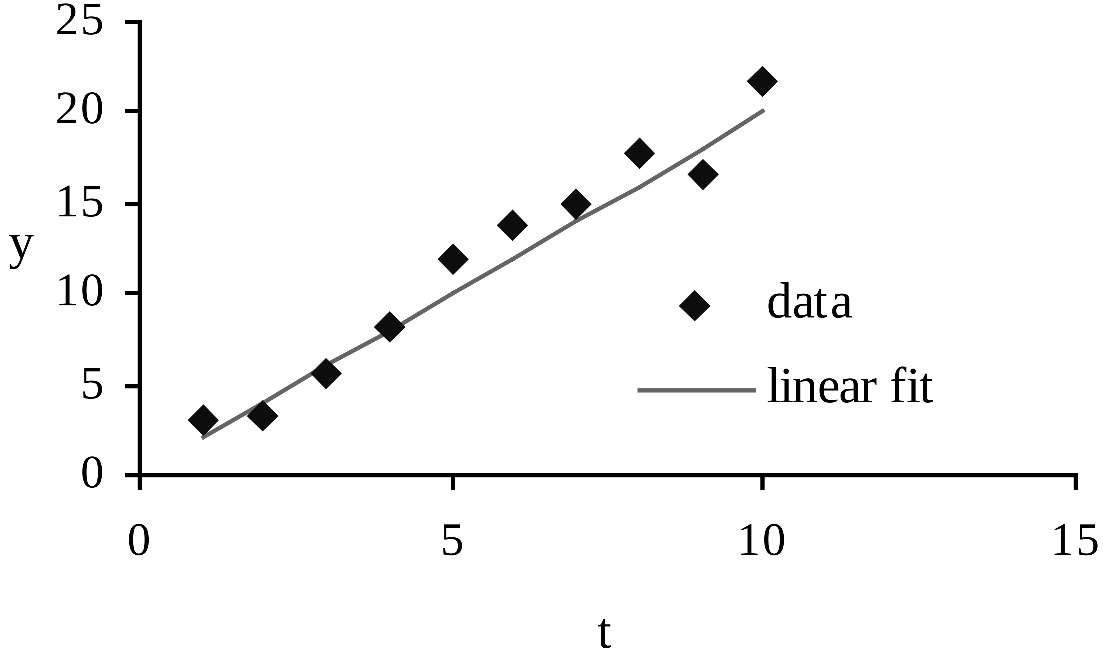
Method of Least Squares



$$J = \sum_{i=1}^{n} \left(m x_i + b - y_i \right)^2$$

The line that results in the minimum value of *J* is the least squares linear fit to the data.

Example

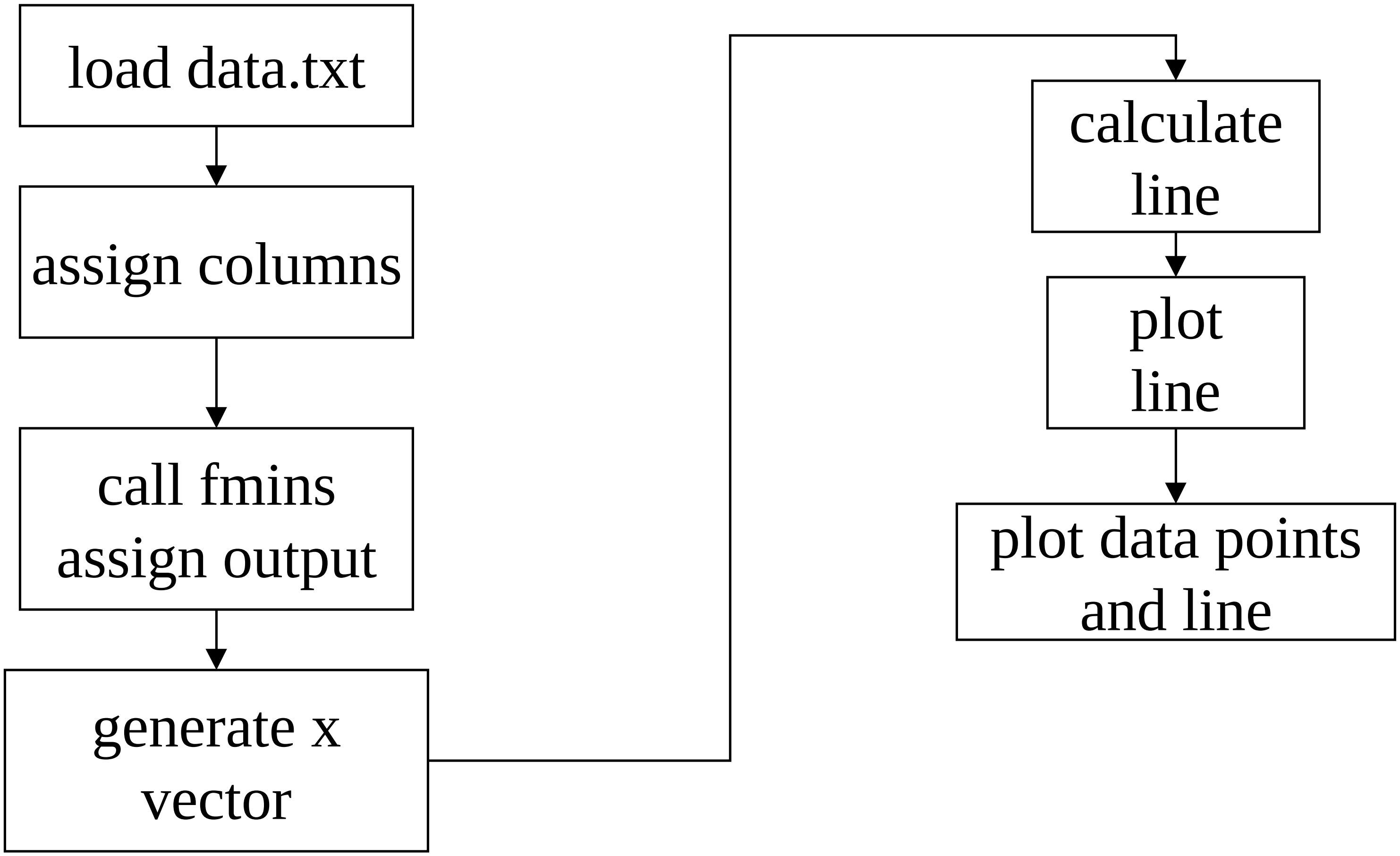


t	data
1	3.00
2	3.16
3	5.66
4	8.15
5	11.84
6	13.85
7	14.88
8	17.67
9	16.53
10	21.65

The Overdetermined System

- Once the curve fit is obtained, a *y*-value may be interpolated for any *x*-value within the *x*-data range (sometimes extrapolation is possible).
- In the following example, fmins is used to minimize the sum of the squared residuals with respect to the slope and intercept.

Flowchart for least.m



Run least.m

Solving The Overdetermined System, Method II

- Solving the overdetermined system is carried out the same way as the other linear algebra solutions using the left division method

$$x = \begin{bmatrix} m \\ b \end{bmatrix} = A \backslash B$$

Method II (cont'd)

- If the system is not overdetermined, the method will not work

Overdetermined system cont.

$$1m + b = 3.00$$

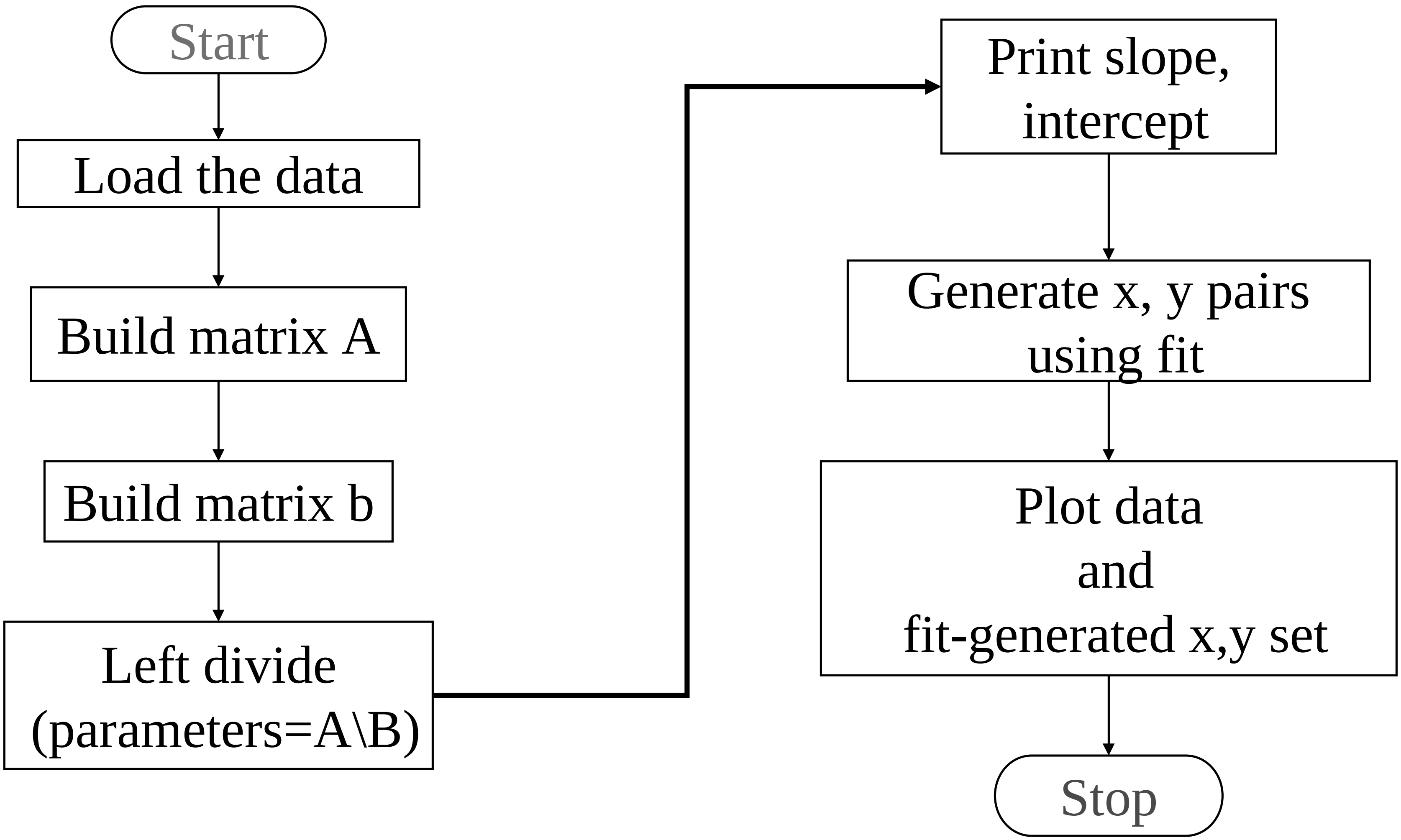
$$5m + b = 11.84$$

$$10m + b = 21.65$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 5 & 1 \\ 10 & 1 \end{bmatrix}}_{\text{A}} \cdot \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\text{x}} = \underbrace{\begin{bmatrix} 3.00 \\ 11.84 \\ 21.65 \end{bmatrix}}_{\text{B}}$$

t	data
1	3.00
2	3.16
3	5.66
4	8.15
5	11.84
6	13.85
7	14.88
8	17.67
9	16.53
10	21.65

Program least2.m



Run least2.m