Projectile Motion

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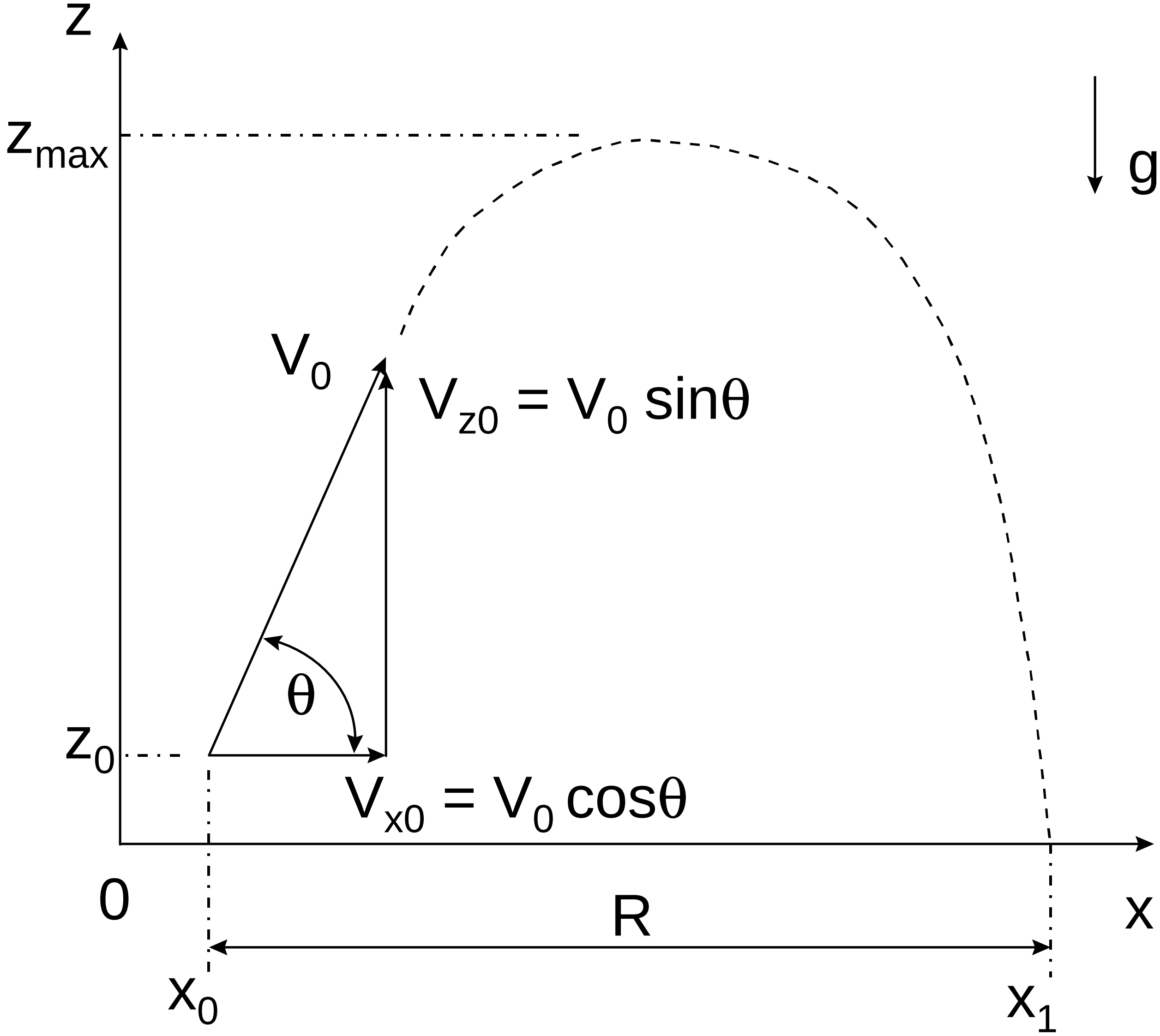
Initial Conditions

- <u>Case</u>: vacuum



Equations of Motion

$$\Sigma F_x = 0 = M (dV_x/dt)$$

$$\Sigma F_z = -Mg = M (dV_z/dt)$$

Integration of Equations

- $V_x(t) = dx/dt = V_0 \cos\theta$
- $x(t) = x_0 + V_{x0}\, t$
- $V_z(t) = dz/dt = V_{z0} - gt$
- $z(t) = z_0 + V_{z0}t - (1/2)gt^2$

Range

When z = 0, x = x_1 and t = t_1. Therefore

$$-\frac{1}{2}gt_1^{\ 2} + V_{z0}t_1 + z_0 = 0$$

$$t_1 = \frac{V_{z0}}{g} + \sqrt{\left(\frac{V_{z0}}{g}\right)^2 + \frac{2z_0}{g}}$$

$$R = x_1 - x_0$$

$$R = V_{x0}t_1$$

Maximum Height

Necessary condition: $\dfrac{dz}{dt} = V_z = 0$

$$t_{\max} = \frac{V_{z0}}{g}$$

$$z_{\max} = z_0 + V_{z0} t_{\max} - \frac{1}{2} g t_{\max}^2$$

$$x_{\max} = x_0 + V_{x0} t_{\max}$$

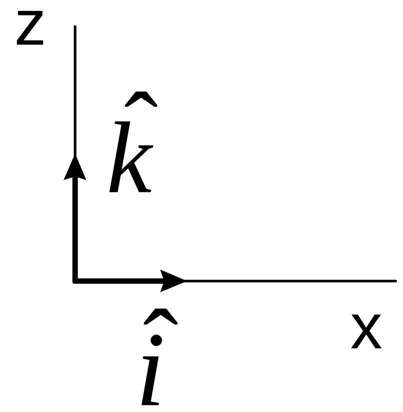
Aerodynamic Drag

$$D = \frac{1}{2} c_D A_D \rho V^2$$

c_D = drag coefficient (non-dimensional)
A_D = reference area (m^2)
ρ = air density (kg/m^3)
V = magnitude of air relative velocity (m/s)

Velocity Magnitude

$$\mathbf{V} = V_x\hat{i} + V_z\hat{k}$$



$$V^2 = \mathbf{V} \cdot \mathbf{V} = V_x^2 + V_z^2$$

$$V = \sqrt{V_x^{\,2} + V_z^{\,2}}$$

Velocity Magnitude Array Method

$$\mathbf{V} = \left[V_x, V_z\right]$$

$$\mathbf{V}.*\mathbf{V} = \left[V_x^2, V_z^2\right]$$

The array product of two vectors is an <u>vector</u>

$$V = sqrt(sum(\mathbf{V}.*\mathbf{V}))$$

Velocity Magnitude Matrix Method

$$\mathbf{V} = \left[V_x, V_z\right]$$

Row Form, 1 row & 2 columns

$$\mathbf{V}' = \begin{bmatrix} V_x \\ V_z \end{bmatrix}$$

Transposed to Column Form, 2 rows & 1 column

$$V^2 = \mathbf{V} * \mathbf{V}'$$

Matrix product

$$V^2 = \left[V_x + V_z\right] \begin{bmatrix} V_x \\ V_z \end{bmatrix}$$

The matrix product of two vectors is a <u>scalar</u>

Matrix Product Rules

- Number of columns of first must equal number of rows of second
- Rule (works for more rows and columns, too)

$$V_{ij}^{2} = \sum_{k=1}^{n_1, m_2} V_{ik} V'_{kj}$$

Range Example

g=9.807 m/s^2, Vz0 = 10 m/s, z_0 = 1.0 m

$$-\frac{1}{2}gt_1^2 + V_{z0}t_1 + z_0 = \begin{bmatrix} -\frac{g}{2} & V_{z0} & z_0 \end{bmatrix} \begin{bmatrix} t_1^2 \\ t_1 \\ 1 \end{bmatrix}$$

$$c = \begin{bmatrix} -\frac{g}{2} & V_{z0} & z_0 \end{bmatrix}$$ MATLAB polynomial

$$Two\_roots = roots(c) = \begin{bmatrix} r_1 & r_2 \end{bmatrix}$$

t_1 equals the physically-meaningful root

$$R = V_{x0}t_1$$

Set path to a:

```
diary        Recording begins
% RANGE CALCULATION
```

Comments like the next one show the structure of the problem

```
% Set initial conditions
```

Define your variables and units

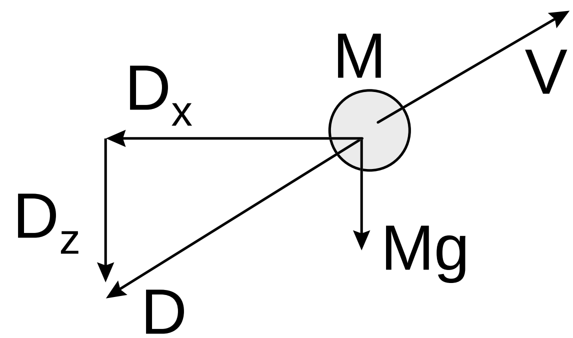
```
g = 9.807;  % grav. accel., m/s^2
V0 = 10.0;  % initial velocity, m/s
theta0d = 45; % launch angle, degrees
z0 = 1.0;          % launch height, m
% Calculate Vz0
theta0 = theta0d*pi/180; Use radians
Vz0 = V0*sin(theta0);
% Polynomial coefficients in array
c = [-.5*g, Vz0, Z0 ];
??? Undefined function or variable 'Z0'.
```

In MATLAB Z0 and z0 aren't the same

```
c = [ -.5*g, Vz0, z0 ];  The highest power comes
first
% Find and display roots of c
Solve -.5*g*t1^2 + Vz0*t1 + z0 = 0
the_roots_of_c = roots(c) Descriptive names help
the_roots_of_c =

    1.5718       the_roots_of_c is a column vector
   -0.1297

% The time of flight (t1) must be positive
t1 = the_roots_of_c(1) Index = 1 selects the first
root
t1 =

    1.5718       Flight time is 1.5718 seconds

% Calculate the range with t1
Vx0 = V0*cos(theta0)  No drag force: Vx0 is
constant
Vx0 =

    7.0711

Range = Vx0*t1
Range =
   11.1143

% The result is in m
diary off                Don't forget to turn it off
```

Projectile with Drag

$$\mathbf{D} = D_x \hat{i} + D_z \hat{k}$$



$$M \frac{dV_z}{dt} = -Mg - D_z$$

$$M \frac{dV_x}{dt} = -D_x$$

Velocity with Drag

$$V_x = V_{x0} - \frac{1}{M}\int_0^t D_x \, dt$$

$$V_z = V_{z0} - gt - \frac{1}{M}\int_0^t D_z \, dt$$

We cannot integrate analytically, so a numerical approximation must be used. More about that later.