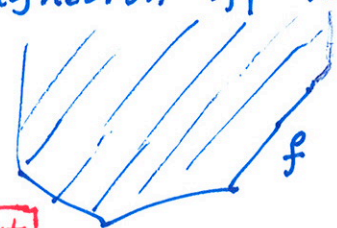
Solutions to Homework 3

1. $\boxed{\text{5 points, first part; 5 points, second part}}$

The epigraph of a function is a halfspace iff that function is affine.

The epigraph of a function is a polyhedron iff that function is piecewise affine.



2. $\boxed{\text{10 points, first part; 5 points, second part}}$

Let $g(t) = f(Z + tV)$, where $Z \succ 0$ and $V \in S^n$. We have

$$\begin{aligned}
g(t) &= \operatorname{tr}\left((Z+tV)^{-1}\right) \\
&= \operatorname{tr}\left(Z^{-1/2}(I + t Z^{-1/2} V Z^{-1/2})^{-1} Z^{-1/2}\right) \\
&= \operatorname{tr}\left(Z^{-1}(I + t Q \Lambda Q^T)^{-1}\right) \\
&= \operatorname{tr}\left(Z^{-1}(QQ^T + t Q\Lambda Q^T)^{-1}\right) \\
&= \operatorname{tr}\left(Z^{-1} Q (I + t\Lambda)^{-1} Q^T\right) \\
&= \operatorname{tr}\left(Q^T Z^{-1} Q (I + t\Lambda)^{-1}\right),
\end{aligned}$$

where $Z^{-1/2} V Z^{-1/2} = Q \Lambda Q^T$. Now, since $I + t\Lambda$ is a diagonal matrix then

$$\operatorname{tr}\left(Q^T Z^{-1} Q (I + t\Lambda)^{-1}\right) = \sum_{i=1}^{n} \left[Q^T Z^{-1} Q\right]_{ii} (1 + t\lambda_i)^{-1}$$

Thus we have

$$g(t) = \sum_{i=1}^{n} \alpha_i (1 + t\lambda_i)^{-1}$$

where $\alpha_i = \left[Q^T Z^{-1} Q\right]_{ii}$. Since $Z \succ 0$ then $Z^{-1} \succ 0$ and thus $Q^T Z^{-1} Q \succ 0$. Therefore $\alpha_i > 0$. Also, since $\operatorname{dom} f = S^n_{++}$ then t and V have to be such that $Z + tV \succ 0$. This implies that $1 + t\lambda_i > 0$. Thus g is the positive weighted sum of the convex functions

$\dfrac{1}{1+t\lambda_i}$, and is thus convex (as a function of t). Since the restriction of f to any line is convex, then f is a convex function.

To prove the convexity of $h(x) = \mathrm{tr}((A_0 + x_1 A_1 + \cdots + x_n A_n)^{-1})$, we note that $h(x)$ can be written as the composition of the (matrix-valued) function $g(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ and the function $f(X) = \mathrm{tr}(X^{-1})$,

$$h(x) = f(g(x))$$
$$= \mathrm{tr}((A_0 + x_1 A_1 + \cdots + x_n A_n)^{-1})$$

Since f is convex and g is affine, it follows that h is convex.

3 $\boxed{\text{10 points}}$

The function f can be written as the pointwise maximum over i of $a_i^T x$,

$$f(x) = \max_i (a_i^T x)$$

where the set $\{a_i\}$ is composed of all possible vectors in $\mathbb{R}^n$ with r elements equal to ± 1 and the rest equal to zero. Since $a_i^T x$ is linear in x for every i, then $f(x)$ is convex.

4 $\boxed{\text{5 points}}$

Since

$$\sum_{i=1}^{k} \lambda_i(X) = \sup\{\mathrm{tr}(V^T X V) \mid V \in \mathbb{R}^{n \times k},\ V^T V = I\}$$

then f is the pointwise supremum of a family of linear (in X) functions $\mathrm{tr}(V^T X V)$. Therefore f is convex.