Solutions to Homework 1 (10 points for each problem)

① Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = 0$.

Now consider a general matrix A, and partition it columnwise

$$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \rightarrow A^T = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix}$$

$$A^T A = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \cdots \\ a_2^T a_1 & \ddots & \vdots \\ \vdots & \cdots & a_n^T a_n \end{bmatrix}.$$

If this matrix is equal to zero, then in particular all of its diagonal entries have to be zero

$$a_1^T a_1 = 0 \rightarrow \|a_1\|_2^2 = 0 \rightarrow a_1 = 0$$

$$\vdots$$

$$a_n^T a_n = 0 \rightarrow \|a_n\|_2^2 = 0 \rightarrow a_n = 0$$

Therefore all columns of A have to equal zero, which means that A is the zero matrix.

② (part a, 2 points; part b, 4 points; part c, 4 points)

ⓐ $X = \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix}$

$$AX = \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

ⓑ $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \rightarrow A\begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix}$

$A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} \rightarrow A\begin{bmatrix}0\\5\\5\end{bmatrix} = \begin{bmatrix}0\\5\\0\end{bmatrix}$

$A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow A\begin{bmatrix}0\\0\\8\end{bmatrix} = \begin{bmatrix}0\\0\\8\end{bmatrix}$

add the left & right hand sides

$$A\underbrace{\left(\begin{bmatrix}3\\3\\3\end{bmatrix} + \begin{bmatrix}0\\5\\5\end{bmatrix} + \begin{bmatrix}0\\0\\8\end{bmatrix}\right)}_{x} = \underbrace{\begin{bmatrix}3\\5\\8\end{bmatrix}}_{b}$$

$\downarrow$

$$x = \begin{bmatrix}3\\8\\16\end{bmatrix}.$$

(c) $A\begin{bmatrix}0\\0\\1\end{bmatrix}$ = third column of $A = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$A\begin{bmatrix}0\\1\\1\end{bmatrix}$ = sum of second & third columns of $A = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$A\begin{bmatrix}1\\1\\1\end{bmatrix}$ = sum of all three columns of $A = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Suppose $A = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}$

$$\begin{cases} a_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \\ a_2 + a_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} \rightarrow a_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow a_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix} \\ a_1 + a_2 + a_3 = \begin{bmatrix}1\\0\\0\end{bmatrix} \rightarrow a_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} \rightarrow a_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix}. \end{cases}$$

$\downarrow$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

③ By definition, $x \in N(A)$. Now partition A row wise

$$A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \end{bmatrix} \rightarrow a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

$$Ax = 0 \rightarrow \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \end{bmatrix} x = 0 \rightarrow \begin{cases} a_1^T x = 0 \\ a_2^T x = 0 \end{cases}$$

This implies that any vector x that is a solution must be orthogonal to both a_1 & a_2.

On the other hand, the set of all $x \in \mathbb{R}^3$, orthogonal to $y \in \mathbb{R}^3$, defines a plane. Therefore

$$\begin{cases} a_1^T x = 0 \rightarrow & x \text{ belongs to the plane } P_1 \text{ orthogonal to } a_1 \\ a_2^T x = 0 \rightarrow & x \text{ belongs to the plane } P_2 \text{ orthogonal to } a_2 \end{cases}$$

→ if x simultaneously satisfies these two equations then it must belong to the intersection of the two planes P_1 & P_2. And the intersection of two distinct planes is a line.

④ It is easy to see that the columns of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ span all of $\mathbb{R}^3$: For example, I can combine the first two columns to get

$$v = a_2 - a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and combine the last two columns to get

$$w = a_3 - a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Together with the first column

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

I obtain the set of vectors $\left\{ \overset{a_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{v}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{w}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$, which clearly spans the whole space.

Thus $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = b$ has a solution for any b.

On the other hand, no matter how I combine the columns of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, I can never create a vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with $b_3 \neq 0$. But if $b_3 = 0$ then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = b$ always has a solution.

⑤ $y(t) = d + ct$

$$\begin{cases} y(-2) = 4 \rightarrow d - 2c = 4 \\ y(-1) = 3 \rightarrow d - c = 3 \\ y(0) = 1 \rightarrow d + 0 = 1 \\ y(2) = 0 \rightarrow d + 2c = 0 \end{cases} \rightarrow \underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} d \\ c \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}}_{v}$$

This is an overdetermined system and has no solution. Therefore we solve the least-squares problem to obtain the best fit

$$x^* = (A^T A)^{-1} A^T v$$

$$= \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

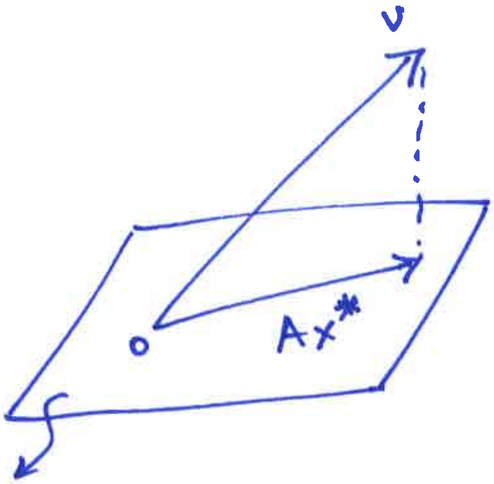
$$= \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$= \frac{1}{36 - 1} \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$= \frac{1}{35} \begin{bmatrix} 61 \\ -36 \end{bmatrix}. \rightarrow$$

best line:

$$y(t) = \frac{61}{35} - \frac{36}{35} t$$



column space of A

$$x^* = (A^TA)^{-1}A^Tv$$

$$\downarrow$$

$$Ax^* = A(A^TA)^{-1}A^Tv$$

projection of v onto the column space of A.

$$Ax^* = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 61 \\ -36 \end{bmatrix} \frac{1}{35}$$

$$= \frac{1}{35} \begin{bmatrix} 133 \\ 97 \\ 61 \\ -11 \end{bmatrix}.$$

(6) First we show that A and A^T have the same (right) null space, i.e., $\mathcal{N}(A) = \mathcal{N}(A^TA)$.

- assume $x \in \mathcal{N}(A)$

$$\Rightarrow Ax = 0 \Rightarrow A^TAx = 0 \Rightarrow x \in \mathcal{N}(A^TA) \qquad (*)$$

- assume $x \in \mathcal{N}(A^TA)$

$$\Rightarrow A^TAx = 0 \Rightarrow x^TA^TAx = 0 \Rightarrow \|Ax\|^2 = 0 \Rightarrow Ax = 0$$

$\Rightarrow x \in \mathcal{N}(A)$ (**)

Now from (*) we conclude that $\mathcal{N}(A) \subseteq \mathcal{N}(A^TA)$ and from (**) we conclude that $\mathcal{N}(A^TA) \subseteq \mathcal{N}(A)$. Therefore we have $\mathcal{N}(A) = \mathcal{N}(A^TA)$.

Now, since the columns of A are linearly independent (this is an assumption we made in the least-squares problem) we have that $\mathcal{N}(A) = 0$. Since $\mathcal{N}(A) = \mathcal{N}(A^TA)$ we get that $\mathcal{N}(A^TA) = 0$. This means that the columns of A^TA are linearly independent. Since A^TA is square and has linearly independent columns it is invertible.