1. [B&V, problem 4.23] Formulate the ℓ_4-norm approximation problem

$$\text{minimize} \quad \|Ax - b\|_4 = (\textstyle\sum_{i=1}^m (a_i^T x - b_i)^4)^{1/4}$$

as a QCQP. The matrix $A \in \mathbb{R}^{m \times n}$ (with rows a_i^T) and the vector $b \in \mathbb{R}^m$ are given.

2. [B&V, problem 4.25] Suppose we are given $K + L$ ellipsoids

$$\mathcal{E}_i = \{P_i u + q_i \mid \|u\|_2 \le 1\}, \quad i = 1, \ldots, K + L,$$

where $P_i \in \mathbb{S}^n$. We are interested in finding a hyperplane that strictly separates $\mathcal{E}_1, \ldots, \mathcal{E}_K$ from $\mathcal{E}_{K+1}, \ldots, \mathcal{E}_{K+L}$, i.e., we want to compute $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ such that

$$a^T x + b \ge 1 \text{ for } x \in \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_K, \qquad a^T x + b \le -1 \text{ for } x \in \mathcal{E}_{K+1} \cup \cdots \cup \mathcal{E}_{K+L},$$

or prove that no such hyperplane exists. Express this problem as an SOCP feasibility problem.

3. [B&V, problem 4.28] In this problem we consider a robust variation of the (convex) quadratic program

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax \preceq b. \end{array}$$

For simplicity we assume that only the matrix P is subject to errors, and the other parameters (q, r, A, b) are exactly known. The robust quadratic program is defined as

$$\begin{array}{ll} \text{minimize} & \sup_{P \in \mathcal{E}} ((1/2)x^T P x + q^T x + r) \\ \text{subject to} & Ax \preceq b \end{array}$$

where $\mathcal{E}$ is the set of possible matrices P.

For each of the following sets $\mathcal{E}$, express the robust QP as a convex problem. Be as specific as you can. If the problem can be expressed in a standard form (e.g., QP, QCQP, SOCP, SDP), say so.

   (a) A finite set of matrices: $\mathcal{E} = \{P_1, \ldots, P_K\}$, where $P_i \in \mathbb{S}^n_+$, $i = 1, \ldots, K$.

   (b) A set specified by a nominal value P_0 plus a bound on the eigenvalues of the deviation $P - P_0$:

$$\mathcal{E} = \{P \in \mathbb{S}^n \mid -\gamma I \preceq P - P_0 \preceq \gamma I\}$$

   where $\gamma \in \mathbb{R}$ and $P_0 \in \mathbb{S}^n_+$.

4. [B&V, problem 4.40] Express the following problems as SDPs.

   (a) The LP

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b. \end{array}$$

   (b) The QP

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

   with $P \succeq 0$, the QCQP

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

   with $P_i \succeq 0$ for $i = 0, 1, \ldots, m$, and the SOCP

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g. \end{array}$$

   Hint: Suppose $A \in \mathbb{S}^r_{++}$, $C \in \mathbb{S}^s$, and $B \in \mathbb{R}^{r \times s}$. Then

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \iff C - B^T A^{-1} B \succeq 0.$$

(c) The matrix fractional optimization problem

$$\text{minimize} \quad (Ax+b)^T F(x)^{-1}(Ax+b)$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$,

$$F(x) = F_0 + x_1F_1 + \cdots + x_nF_n,$$

with $F_i \in \mathbb{S}^m$, and we take the domain of the objective to be $\{x \mid F(x) \succ 0\}$. You can assume the problem is feasible (there exists at least one x with $F(x) \succ 0$).

5. [B&V, problem 4.43] Suppose $A : \mathbb{R}^n \to \mathbb{S}^m$ is affine, i.e.,

$$A(x) = A_0 + x_1A_1 + \cdots + x_nA_n$$

where $A_i \in \mathbb{S}^m$. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$ denote the eigenvalues of $A(x)$. Show how to pose the following problems as SDPs.

(a) Minimize the maximum eigenvalue $\lambda_1(x)$.

(b) Minimize the spread of the eigenvalues, $\lambda_1(x) - \lambda_m(x)$.