1. A linear program (LP) is said to be in *standard form* if the only inequalities are componentwise nonnegativity constraints $x \succeq 0$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0. \end{array}$$

It is sometimes useful to transform a general LP

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

to one in standard form (for example, in order to use an algorithm for solving standard form LPs). Convert the general LP given above into an equivalent standard form LP.

Hint: Express x as the difference of two nonnegative variables, $x = x^+ - x^-$ with $x^+, x^- \succeq 0$.

2. [B&V, problem 4.11] Formulate the following problems as LPs.

(a) Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$.

(b) Minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$.

(c) Minimize $\|Ax - b\|_1 + \|x\|_\infty$.

In each problem, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given.

3. [B&V, problem 4.13] Consider the problem, with variable $x \in \mathbb{R}^n$,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \text{ for all } A \in \mathcal{A}, \end{array}$$

where $\mathcal{A} \subseteq \mathbb{R}^{m \times n}$ is the set

$$\mathcal{A} = \{A \in \mathbb{R}^{m \times n} \mid \bar{A}_{ij} - V_{ij} \leq A_{ij} \leq \bar{A}_{ij} + V_{ij},\ i = 1, \ldots, m,\ j = 1, \ldots, n\}.$$

(The matrices $\bar{A}$ and V are given.) This problem can be interpreted as an LP where each coefficient of A is only known to lie in an interval, and we require that x must satisfy the constraints for all possible values of the coefficients.

Express this problem as an LP. The LP you construct should be efficient, i.e., it should not have dimensions that grow exponentially with n or m.

4. [B&V, problem 4.15] In a *Boolean linear program*, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0, 1\},\ i = 1, \ldots, n. \end{array} \tag{1}$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).

In a general method called *relaxation*, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \leq x_i \leq 1$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1,\ i = 1, \ldots, n. \end{array} \tag{2}$$

We refer to this problem as the *LP relaxation* of the Boolean LP (1). The LP relaxation is far easier to solve than the original Boolean LP.

(a) Show that the optimal value of the LP relaxation (2) is a lower bound on the optimal value of the Boolean LP (1). What can you say about the Boolean LP if the LP relaxation is infeasible?

(b) It sometimes happens that the LP relaxation has a solution with $x_i \in \{0, 1\}$. What can you say in this case?

5. [B&V, problem 4.19] Consider the problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_1/(c^T x + d) \\ \text{subject to} & \|x\|_\infty \leq 1, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $d \in \mathbb{R}$. We assume that $d > \|c\|_1$, which implies that $c^T x + d > 0$ for all feasible x.

(a) Show that this is a quasiconvex optimization problem. [A quasiconvex optimization problem is one in which the inequality constraint functions are convex, the equality constraint functions are affine, and the objective function is quasiconvex.]

(b) Show that it is equivalent to the convex optimization problem

$$\begin{array}{ll} \text{minimize} & \|Ay - bt\|_1 \\ \text{subject to} & \|y\|_\infty \leq t \\ & c^T y + dt = 1, \end{array}$$

with variables $y \in \mathbb{R}^n$, $t \in \mathbb{R}$.