1. [B&V, problem 3.14] We say the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is *convex-concave* if $f(x, z)$ is a concave function of z, for each fixed x, and a convex function of x, for each fixed z. We also require its domain to have the product form $\mathbf{dom}\, f = A \times B$, where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are convex.

   (a) Give a second-order condition for a twice differentiable function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ to be convex-concave, in terms of its Hessian $\nabla^2 f(x, z)$.

   (b) Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex-concave and differentiable, with $\nabla f(\tilde{x}, \tilde{z}) = 0$. Show that the *saddle-point property* holds: for all x, z, we have

   $$f(\tilde{x}, z) \le f(\tilde{x}, \tilde{z}) \le f(x, \tilde{z}).$$

   (This can be used to show that f satisfies the *strong max-min property*:

   $$\sup_z \inf_x f(x, z) = \inf_x \sup_z f(x, z),$$

   with the common value being $f(\tilde{x}, \tilde{z})$.)

   (c) Now suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is differentiable, but not necessarily convex-concave, and the saddle-point property holds at $\tilde{x}, \tilde{z}$:

   $$f(\tilde{x}, z) \le f(\tilde{x}, \tilde{z}) \le f(x, \tilde{z})$$

   for all x, z. Show that $\nabla f(\tilde{x}, \tilde{z}) = 0$.

2. [B&V, problem 3.22] Use the composition rules to show that the following function is convex.

   $$f(x, u, v) = -\log(uv - x^T x) \quad \text{on} \quad \mathbf{dom}\, f = \{(x, u, v) \mid uv > x^T x,\ u,\ v > 0\}.$$

   Hint: Use the convexity of the quadratic-over-linear function.

3. [B&V, problem 3.32] In general the product or ratio of two convex functions is not convex. However, there are some results that apply to functions on $\mathbb{R}$. Prove the following.

   (a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.

   (b) If f and g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.

   (c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then f/g is convex.

4. [B&V, problem 4.1] Consider the optimization problem

   $$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \ge 1 \\ & x_1 + 3x_2 \ge 1 \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{array}$$

   Make a sketch of the feasible set. For each of the following objective functions, give an optimal point x^* and the optimal value $p^* = f_0(x^*)$. [In some cases the optimal value may be achieved by a set of points rather than a single point; in such cases determine the optimal *set*.]

   (a) $f_0(x_1, x_2) = x_1 + x_2$.

   (b) $f_0(x_1, x_2) = -x_1 - x_2$.

   (c) $f_0(x_1, x_2) = x_1$.

   (d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$.

   (e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

5. [B&V, problem 4.6] A convex optimization problem can have only *linear* equality constraint functions. In some special cases, however, it is possible to handle convex equality constraint functions, i.e., constraints of the form $g(x) = 0$, where g is convex. We explore this idea in this problem.

   Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h(x) = 0, \end{array} \tag{1}$$

   where f_i and h are convex functions with domain $\mathbb{R}^n$. Unless h is affine, this is *not* a convex optimization problem. Consider the related problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h(x) \leq 0, \end{array} \tag{2}$$

   where the convex equality constraint has been relaxed to a convex inequality. [A change in a constraint is said to be a relaxation if it enlarges the constraint set.] This problem is, of course, convex.

   Now suppose we can guarantee that at any optimal solution x^* of the convex problem (2), we have $h(x^*) = 0$, i.e., the inequality $h(x) \leq 0$ is always active at the solution. Then we can solve the (nonconvex) problem (1) by solving the convex problem (2).

   Show that this is the case if there is an index r such that

   - f_0 is monotonically increasing in x_r
   - $f_1, \ldots, f_m$ are nondecreasing in x_r
   - h is monotonically decreasing in x_r.