1. [B&V, problem 3.6] When is the epigraph of a function a halfspace? When is the epigraph of a function a polyhedron?

2. [B&V, problems 3.18,20] Adapt the proof of convexity of the negative log-determinant function discussed in class to show that $f(X) = \mathrm{trace}(X^{-1})$ is convex on $\mathrm{dom}\, f = \mathbf{S}^n_{++}$. Use this to prove the convexity of $h(x) = \mathrm{trace}(A_0 + x_1 A_1 + \cdots + x_n A_n)^{-1}$ on the set $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$ with $A_i \in \mathbf{S}^m$.

3. [B&V, problem 3.21] Show that $f(x) = \sum_{i=1}^{r} |x|_{[i]}$ is convex on $\mathbb{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (i.e., $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$. In other words, $|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.

4. [B&V, problem 3.26] Let $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ denote the eigenvalues of a matrix $X \in \mathbf{S}^n$. We have already seen several functions of the eigenvalues that are convex (or concave) functions of X.

   - The maximum eigenvalue $\lambda_1(X)$ is convex. The minimum eigenvalue $\lambda_n(X)$ is concave.
   - The sum of the eigenvalues, which is also equal to the trace of X, $\mathrm{trace}(X) = \sum_{i=1}^{n} \lambda_i(X)$, is linear.
   - The sum of the inverses of the eigenvalues, which is equal to the trace of the inverse of X, $\mathrm{trace}(X^{-1}) = \sum_{i=1}^{n} 1/\lambda_i(X)$, is convex on $\mathbf{S}^n_{++}$.
   - The negative logarithm of the product of the eigenvalues, $-\log\det X = -\sum_{i=1}^{n} \log \lambda_i(X)$, is convex on $\mathbf{S}^n_{++}$.

   Show that the sum of the k largest eigenvalues, $f(X) = \sum_{i=1}^{k} \lambda_i(X)$, is convex on $\mathbf{S}^n$.

   Hint: Use the characterization $\sum_{i=1}^{k} \lambda_i(X) = \sup\{\mathrm{trace}(V^T X\, V) \mid V \in \mathbb{R}^{n\times k},\ V^T V = I\}$.