1. Show that $A^2 = 0$ is possible for a nonzero matrix A, but that $A^TA = 0$ is only possible if $A = 0$.

2. Suppose you solve $Ax = b$ for three special right-hand sides b,

$$Ax_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad Ax_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad Ax_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

   (a) If the solutions x_1, x_2, x_3 are the columns of a matrix X, what is AX?

   (b) If the three solutions in Part (a) are

$$x_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad x_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad x_3 = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

   solve $Ax = b$ when $b = [3, 5, 8]^T$.

   (c) Given the information in Parts (a) and (b), what is A?

3. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix}1 & 1 & 1\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

   form (a) a plane; (b) a line; (c) a point; (d) the null space of A; (e) the column space of A.

4. For which vectors $[b_1, b_2, b_3]^T$ do these systems have a solution?

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}, \quad \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}.$$

5. Find the best straight-line fit (in the sense of least-squares) to the measurements

$$\begin{array}{llll} y = 4 \text{ at } t = -2, & & y = 3 \text{ at } t = -1, \\ y = 1 \text{ at } t = 0, & & y = 0 \text{ at } t = 2. \end{array}$$

   Then find the projection of $y = [4, 3, 1, 0]^T$ onto the column space of

$$A = \begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 2\end{bmatrix}.$$

6. In solving the least squares problem in class, we *assumed* that A^TA is invertible. Prove that A^TA is indeed invertible, i.e., using the assumption made in the least squares problem that the columns of A are linearly independent, prove that $(A^TA)^{-1}$ exists.

   Hint: (a) Show that the two matrices A and A^TA have the same null space, i.e., $\mathcal{N}(A) = \mathcal{N}(A^TA)$; (b) Use the assumption that the columns of A are linearly independent and therefore its null space contains only the vector zero; (c) Use the fact that if the null space of a square matrix contains only the zero vector, then that matrix is invertible.