high level objectives of this course

- recognize and formulate convex optimization problems: requires learning about

  * convex sets
  * convex functions
  * equivalent formulations

- identify approximations and relaxations that render a non-convex problem convex

- understand numerical algorithms and software that solve optimization problems

review of linear algebra

- a matrix is an array of numbers with m rows & n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$



- for two matrices A, B, assuming they have the same dimension, we define.

$$A + B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

$$A + B = B + A \qquad A + (B + C) = (A + B) + C$$

- the "transpose" of a matrix is defined as

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & \cdot & & \\ \vdots & & & \\ a_{1n} & & & a_{mn} \end{bmatrix}$$

A : $m \times n$

A^T : $n \times m$

$$A = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \qquad A^T = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad (A^T)^T = A.$$

(A^H or A^* is $\overline{(A^T)}$ ← complex conjug.)

- a square matrix is "symmetric" if $A^T = A$.

- example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A^T \neq A$$

main diagonal

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad A^T = A.$$

- if matrix is multiplied by scalar, then all its entries are multiplied.
- if A is an $n \times m$ matrix and B is an $m \times p$ matrix, then $C = AB$ is an $n \times p$ matrix defined as

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$C =$ [A] [B]

($n \times m$ for A; $m \times p$ for B; m connects them)

(# of col. of A = # of rows of B.)

- example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 2 \end{bmatrix}_{3\times 2} \qquad B = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}_{2\times 2}.$$

(BA makes no sense)

$$C = AB = \begin{bmatrix} (1)(3)+(2)(-1) & 4 \\ 5 & 5 \\ -5 & 0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 4 \\ 5 & 5 \\ -5 & 0 \end{bmatrix}$$

- in general $AB \neq BA$, but $(AB)C = A(BC)$ always.

- if a matrix has only one column, then it is called a "column vector". (n-vector)

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$ $n \times 1$ dimension.

- if r, q are two vectors of equal dimension, we define their "inner product" as

$$\langle r, q \rangle = \langle q, r \rangle \quad (= r \cdot q = q \cdot r)$$

$$= r^T q = q^T r$$

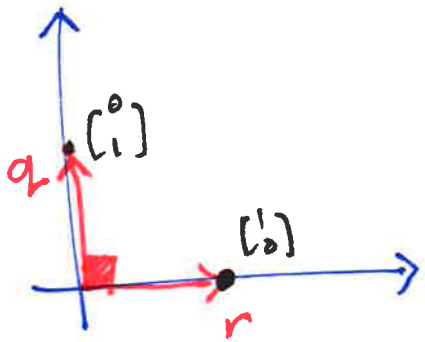
$$= \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = r_1 q_1 + r_2 q_2 + \cdots + r_n q_n$$

$$= \sum_{i=1}^{n} r_i \cdot q_i$$

the result of an inner product is always a scalar.

- r and q are "orthogonal" if $r^T q = 0$

- example: $r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



$$r^T q = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \cdot 0 + 0 \cdot 1$$
$$= 0$$

- the "norm" of a vector r is defined as

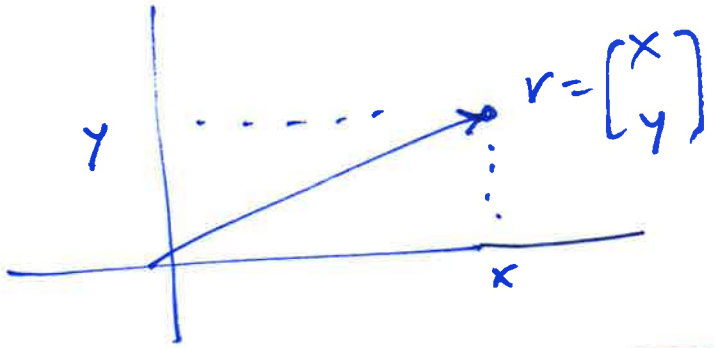
$$\|r\|^2 = r^T r = \sum_{i=1}^{n} r_i^2 = r_1^2 + \cdots + r_n^2$$

$$\left(\|r\| = \sqrt{r_1^2 + \cdots + r_n^2} \right)$$

$$\|r\| = 0 \iff r = 0_{n \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- the norm of a vector can be interpreted as its "length".

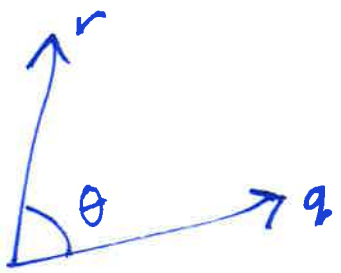
in 2 dimensions



$$\|r\| = \sqrt{x^2+y^2} = \text{length of vector } r.$$

- we have the following

$$r^T q = \|r\| \cdot \|q\| \cdot \cos(\theta).$$



$$r^T r = \|r\| \|r\| \cos 0 = \|r\|^2$$

$$r \perp q \rightarrow r^T q = \|r\| \|q\| \cos\frac{\pi}{2} = 0.$$

- interpretation 1 of matrix multiplication

$A: n \times m \qquad B: m \times p$

$$AB = \begin{bmatrix} \text{---} & a_1^T & \text{---} \\ \text{---} & a_2^T & \text{---} \\ & \vdots & \\ \text{---} & a_n^T & \text{---} \end{bmatrix}_{n \times m} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix}_{m \times p}$$

$$= \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & & \\ \vdots & & \ddots & \\ a_n^T b_1 & & & a_n^T b_p \end{bmatrix}_{n \times p}$$

(this is how we usually multiply matrices anyway!)

- example: (interp. 1)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 2 \end{bmatrix} \quad a_1^T, \; a_2^T, \; a_3^T \qquad B = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \quad b_1, \; b_2$$

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} & \begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 5 & 5 \\ -5 & 0 \end{bmatrix}.$$

- interpretation 2 of matrix multiplication

$A: n \times m \qquad B: m \times p$

$$AB = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} — & b_1^T & — \\ — & b_2^T & — \\ & \vdots & \\ — & b_m^T & — \end{bmatrix}$$

$n \times m$ $\qquad$ $m \times p$

$a_1: n \times 1$

$b_1^T: 1 \times p$

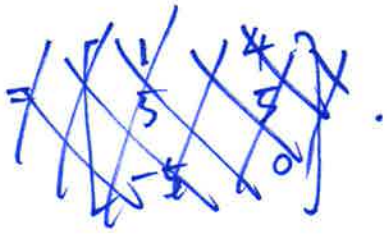
$a_1 b_1^T: n \times p$

$$= \underbrace{a_1 b_1^T}_{n \times p} + \underbrace{a_2 b_2^T}_{n \times p} + \cdots + \underbrace{a_m b_m^T}_{n \times p}$$

- example: (interp. 2)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 2 \end{bmatrix} \quad (a_1, a_2) \qquad B = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \quad (b_1^T, b_2^T)$$

$$AB = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_{3\times 1} \begin{bmatrix} 3 & 2 \end{bmatrix}_{1\times 2} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix}$$



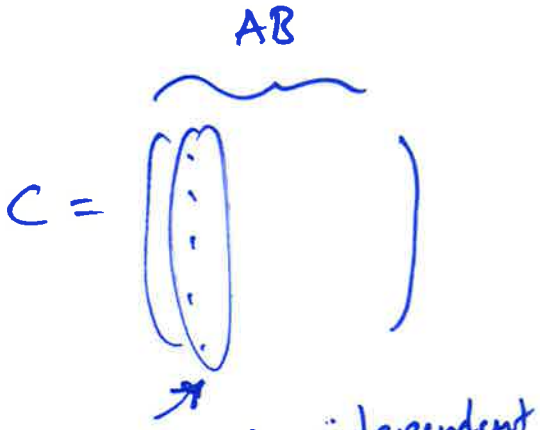
$$= \begin{bmatrix} 3 & 2 \\ 6 & 4 \\ -3 & -2 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 5 & 5 \\ -5 & 0 \end{bmatrix}.$$

- interpretation 3 of matrix multiplication

$$AB = A_{n\times m}\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix}_{m\times p}$$

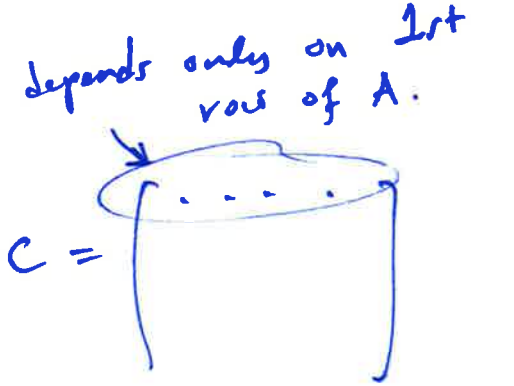
$$= \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{bmatrix}_{n\times p}$$



- interpretation 4 of matrix multiplication

$$AB = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix}_{n\times m} B_{m\times p}$$

$$= \begin{bmatrix} — & a_1^T B & — \\ — & a_2^T B & — \\ & \vdots & \\ — & a_n^T B & — \end{bmatrix}_{n\times p}$$



- example : (interp. 3)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} \overbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}}^{Ab_1} & \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{Ab_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 5 & 5 \\ -5 & 0 \end{bmatrix}$$

- example : (interp. 4)

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \end{bmatrix} \begin{matrix} \} a_1^T B \\ \} a_2^T B \\ \} a_3^T B \end{matrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 5 & 5 \\ -5 & 0 \end{bmatrix}$$

- the identity matrix $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{(\text{square})}$, has the property that for any matrices (& vectors) X, Y

$$I_{m\times m} X_{m\times n} = X_{m\times n}$$

$$Y_{m\times n} I_{n\times n} = Y_{m\times n}$$

(the dimension of I has to be adjusted)

- the square matrix X is the "inverse" of the square matrix A if

$$XA = AX = I$$

then we use notation $A^{-1} := X$

aside: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \neq \begin{bmatrix} 1 & 1/2 \\ 1/3 & 1/4 \end{bmatrix}$.

- example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

aside (on matrix inverses)

- if A is not square, A^{-1} does not make sense
  (→ we have to consider "pseudo-inverses")

- even if A is square, A^{-1} may still not exist
  (→ a pseudo-inverse exists, however)

- even if A is square & invertible, A^{-1} may be highly unreliable, due to sensitivity to numerical errors.
  (→ the "condition number" of A warns us of this)

- if A is square & invertible with a "reasonable" condition number, then A^{-1} can be computed (expensive) & used.

- if the entries of A are functions of an indendent variable t,

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots \\ a_{21}(t) & a_{22}(t) & \cdots \\ \vdots & \vdots & \ddots \\ & & a_{nm}(t) \end{bmatrix}$$

then the derivative of $A(t)$ is defined as

$$\frac{d}{dt} A(t) = \dot{A}(t) = \begin{bmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \cdots \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

- it is not difficult to show that

$$\frac{d}{dt}\left[A(t)\ B(t) \right] = \dot{A}(t)\ B(t) + A(t)\ \dot{B}(t).$$

- integral of $A(t)$

$$\int^{\tau} A(t)\,dt = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \int^{\tau} a_{ij}(t)\,dt & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

- Laplace (Fourier) transform

$$\mathcal{L}\{A(t)\} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathcal{L}\{a_{ij}(t)\} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

- word of caution:

$$A^2 \neq \begin{bmatrix} \cdot & a_{ij}^2 & \cdot \end{bmatrix}, \quad A^{-1} \neq \begin{bmatrix} \cdot & \frac{1}{a_{ij}} & \cdot \end{bmatrix}, \quad e^{A} \neq \begin{bmatrix} \cdot & e^{a_{ij}} & \cdot \end{bmatrix}$$

- we say that v is a "linear combination" of the vectors $v_1, \dots, v_n$ if there exist coefficients $\underbrace{\beta_1, \dots, \beta_n}_{\in \mathbb{R}}$ such that

$$v = \beta_1 v_1 + \cdots + \beta_n v_n.$$

- example: $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a linear combination of

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$v = 1 \cdot v_1 + 2 \cdot v_2.$$

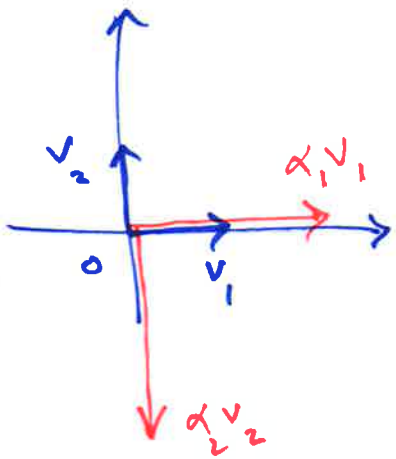
linear independence

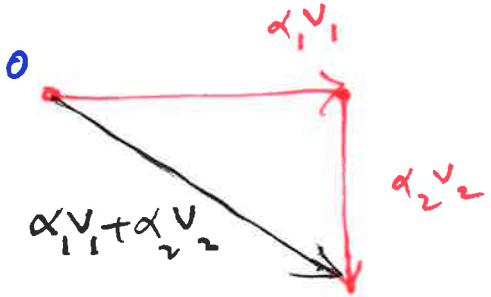
- we say that the vectors $v_1, \dots, v_n$ are "linearly independent" if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \qquad \alpha_1, \dots, \alpha_n \in \mathbb{R}$$

is possible only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ (otherwise the vectors are said to be "linearly dependent")

- example: $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.





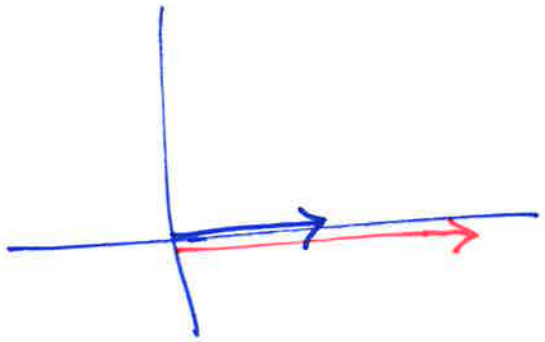
$$\alpha_1 v_1 + \alpha_2 v_2 = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Downarrow$$

$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$$

linearly indep.

- example: $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.



$$\alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\alpha_1 + 2\alpha_2 = 0 \rightarrow \begin{cases} \alpha_1 = 2 \\ \alpha_2 = -1 \end{cases}, \begin{cases} \alpha_1 = -4 \\ \alpha_2 = 2 \end{cases}, \dots$$

linearly dep.

$v_1 = \frac{1}{2} v_2$

$1 \cdot v_1 - \frac{1}{2} v_2 = 0$ ($1 \rightarrow \alpha_1$, $-\frac{1}{2} \rightarrow \alpha_2$)

- example: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$1 \cdot v_1 + 2 \cdot v_2 = v_3$$

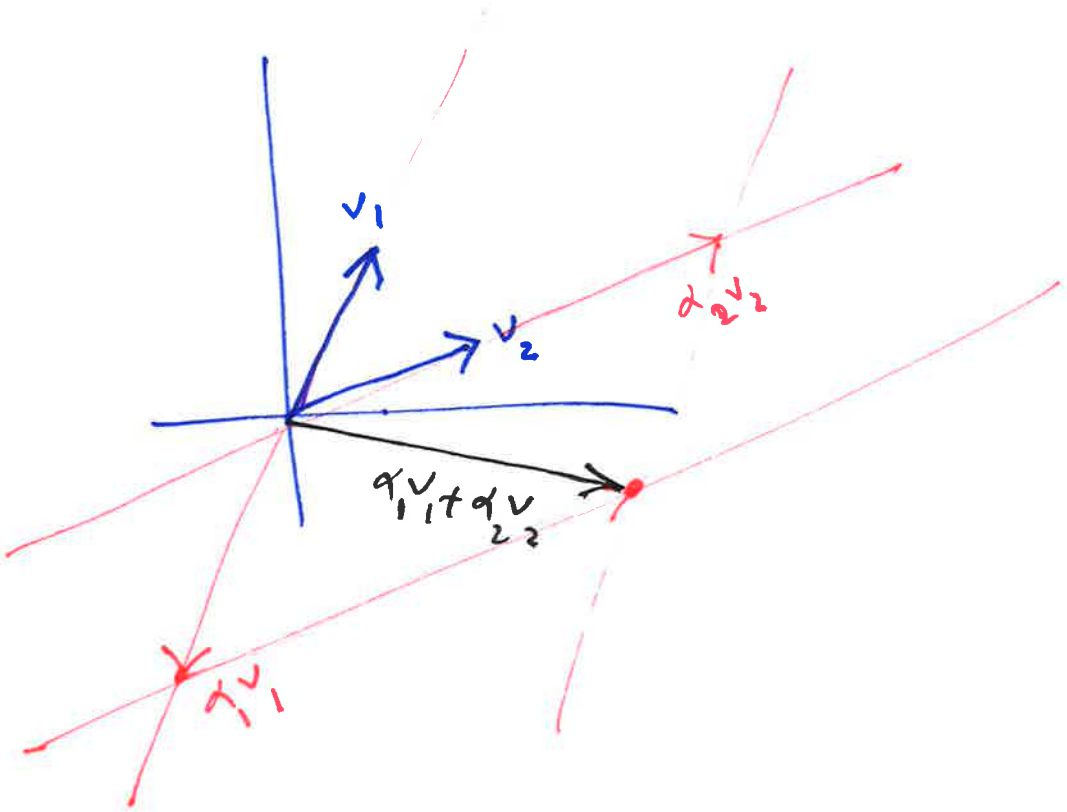
$$1 \cdot v_1 + 2 \cdot v_2 - 1 \, v_3 = 0$$

($1 \rightarrow \alpha_1$, $2 \rightarrow \alpha_2$, $-1 \rightarrow \alpha_3$)

linearly dep.

(if you can write some vector as lin comb. of others, then they are linearly dep.)

- example: $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.



$$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \end{cases}$$

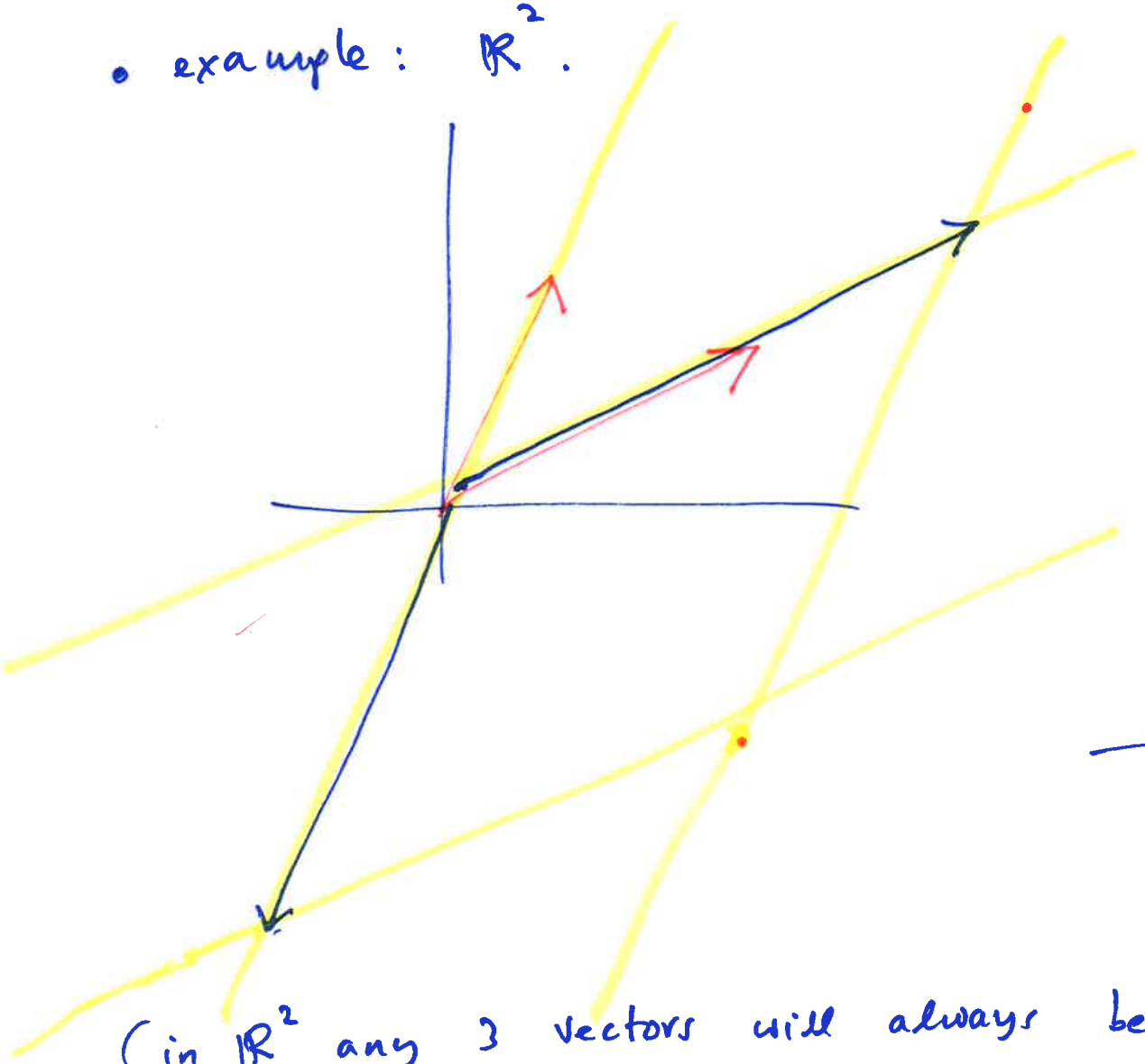
$$\rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$$

- we say $v_1, \ldots, v_n$ are "linearly dependent" if they are not linearly independent.

linearly indep.

- the maximum number of linearly independent vectors in a space is called the "dimension" of that space

- example: $\mathbb{R}^2$.



$$v_3 = \alpha_1 v_1 + \alpha_2 v_2$$

$$\alpha_1 v_1 + \alpha_2 v_2 - v_3 = 0$$

$\rightarrow d = 2.$

(in $\mathbb{R}^2$ any 3 vectors will always be linearly dependent)

- if the dimension of a space is n, and if $v_1, \ldots, v_n$ are linearly independent, then every vector in that space can be written as a linear combination of $v_1, \ldots, v_n$.

  i.e., for any y there exist $\alpha_1, \ldots, \alpha_n$ such that

$$y = \alpha_1 v_1 + \cdots + \alpha_n v_n .$$

→ in this case, the set of vectors $\{v_1, \ldots, v_n\}$ is called a "basis" for this space.

- example: space is $\mathbb{R}^n$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \quad \cdots, \quad v_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}.$$

$$y = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

- an "orthonormal" basis is a basis in which all vectors are mutually orthogonal & have unit length.

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}, \quad v_i \perp v_j \quad i \neq j \quad (v_i^T v_j = 0),$$

basis

$$\|v_i\| = 1 \quad \forall i \quad (v_i^T v_i = 1)$$

- example: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- example: $w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\rightarrow$ $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- example: Fourier analysis. (cos & sin functions are the basis elements)

- if $v_1, \ldots, v_n$ are linearly independent in n-dimensional space (i.e., they form a basis), then any vector y is "uniquely" representable as a linear combination of $v_1, \ldots, v_n$.

proof by contradiction:

$$\begin{cases} y = \alpha_1 v_1 + \cdots + \alpha_n v_n \\ y = \beta_1 v_1 + \cdots + \beta_n v_n \end{cases} \quad \& \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \neq \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n$$

$$(\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_n - \beta_n) v_n = 0.$$

from linear independence, we have

$$\begin{cases} \alpha_1 - \beta_1 = 0 \\ \vdots \\ \alpha_n - \beta_n = 0 \end{cases} \longrightarrow \begin{cases} \alpha_1 = \beta_1 \\ \vdots \\ \alpha_n = \beta_n \end{cases}$$

which is a contradiction.

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fundamental subspaces of a matrix

motivation : consider solving the linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = y_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = y_n \end{cases}$$

$x_1, \ldots, x_m$ unknown

a_{ij} & $y_1, \ldots, y_n$ known

$$\Updownarrow$$

$$Ax = y$$

A : $n \times m$ matrix

x : $m \times 1$ vector

y : $n \times 1$ vector

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}_{m \times 1}$$

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_{n \times 1}$$

we know A & y, we seek x.

- we would like to use linear algebra to answer the following questions

  - when does $Ax = y$ have a solution?
  - when is the solution unique?
  - (later: when no solution, what else can we say?)

- very useful to think of $Ax = y$ as

$$Ax = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \qquad x_i \in \mathbb{R}$$

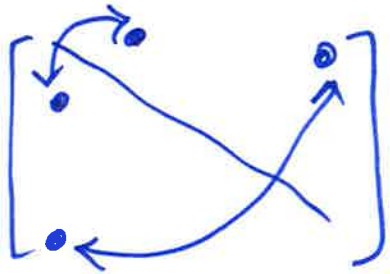
$$= x_1 \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} + \cdots + x_m \begin{bmatrix} | \\ a_m \\ | \end{bmatrix}$$

(interp 2 of matrix multiplication!)

$$= y$$

i.e., Ax gives linear combination of columns of A (may or may not equal y.)

review of last lecture



- vectors, matrices, symmetric matrices
- inner product $\langle r, q \rangle = r^T q = \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \vdots \\ \cdot \end{bmatrix}$
- orthogonality $r^T q = 0$
- norm (or length) of a vector $\|r\|^2 = r^T r$

  $(\|r\| = 0 \iff r = 0)$
- $C = AB$ $\quad A: n \times m \quad B: m \times p$
- $AB \neq BA$ in general.

  (BA make not even make sense)
- X inverse of A if $XA = AX = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

review of last lecture

- $C = AB$ $\quad A: n \times m \quad B: m \times p$ $\quad AB \neq BA$.

- $AB = \begin{bmatrix} - & a_1^T & - \\ & A & \\ & \rightarrow & \end{bmatrix} \begin{bmatrix} | & & | \\ b_1 & B & \\ | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & \cdots \\ \vdots & \ddots \end{bmatrix}$ interp. 1

$AB = \begin{bmatrix} | & | & \\ a_1 & A & \\ | & | & \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & B & \\ & \rightarrow & \end{bmatrix} = \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \end{bmatrix} + \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} + \cdots$ interp. 2

$AB = A \begin{bmatrix} | & & | \\ b_1 & B & \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Ab_1 & \cdots & Ab_p \\ | & & | \end{bmatrix}$ interp. 3

$AB = \begin{bmatrix} - & a_1^T & - \\ & A & \\ & \rightarrow & \end{bmatrix} B = \begin{bmatrix} - & a_1^T B & - \\ & \vdots & \\ - & a_n^T B & - \end{bmatrix}$ interp. 4

review of last lecture

- linear independence

$v_1, \cdots, v_n$ linear independence $\iff$ $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ if $\alpha_1 = \cdots = \alpha_n = 0$

linear combination.

- dim. of space = max. # of linear independent vectors

- if $v_1, \cdots, v_n$ are linearly independent in n-dimensional space, then every vector in that space can be written as a linear combination of $v_1, \cdots, v_n$.

$\{v_1, \cdots, v_n\}$ is a "basis"

review of last lecture

- orthonormal basis: set of basis vectors $\{v_1, \cdots, v_n\}$ s.t.

$$v_i \perp v_j \quad i \neq j$$

$$\|v_i\| = 1 \quad \forall i$$

- uniqueness of basis representation of a vector

$$y = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

α_i are unique.

$$y^{(v)} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \rightarrow \text{in the } \{v_1, \cdots, v_n\} \text{ basis.}$$

in $\mathbb{R}^n$: $v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\cdots$, $v_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

$$y = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

- $Ax = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$

$= x_1 \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} + \cdots + x_m \begin{bmatrix} | \\ a_m \\ | \end{bmatrix}$

> Ax gives linear comb. of columns of A.

- $Ax = y \longrightarrow$
  - any sol ?
  - unique sol ?
  - if no sol, then what ???

2-5

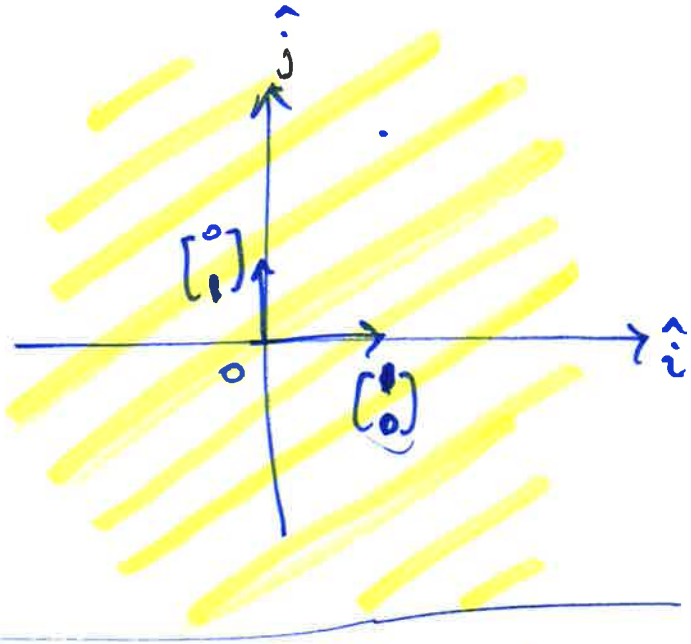
- as x varies over all possible vectors in $\mathbb{R}^n$, we call the resulting set of vectors Ax the "column space" of A

$$\mathcal{C} = \{ y \text{ such that } y = Ax, \; x \in \mathbb{R}^n \}.$$

(also referred to as "range space")

- example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

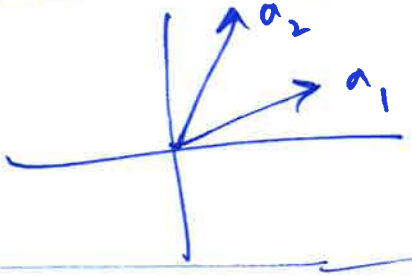
$$Ax = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



$$\rightarrow \quad \mathcal{C} = \mathbb{R}^2.$$

---

aside 1 :



$$\rightarrow \quad \mathcal{C} = \mathbb{R}^2 \qquad A = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$$

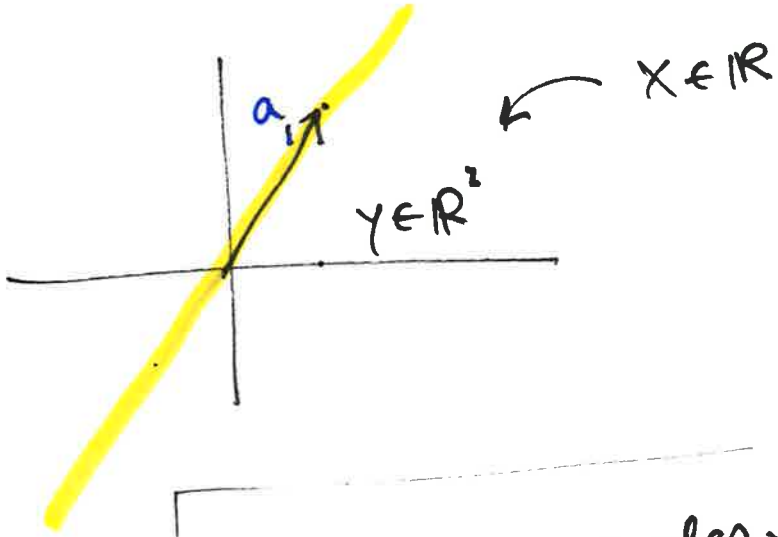
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aside 2 : if $x_1 \geq 0$, $x_2 \geq 0$ $\longrightarrow$



- example: $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$Ax = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{a_1} x \qquad x \in \mathbb{R}$$



in both examples:

$$\mathcal{C} = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \alpha \in \mathbb{R} \right\}.$$

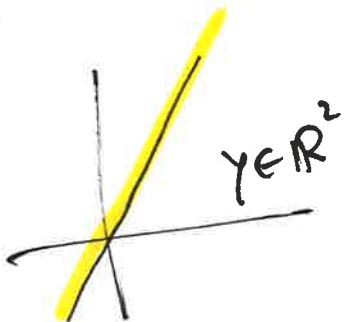
- example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

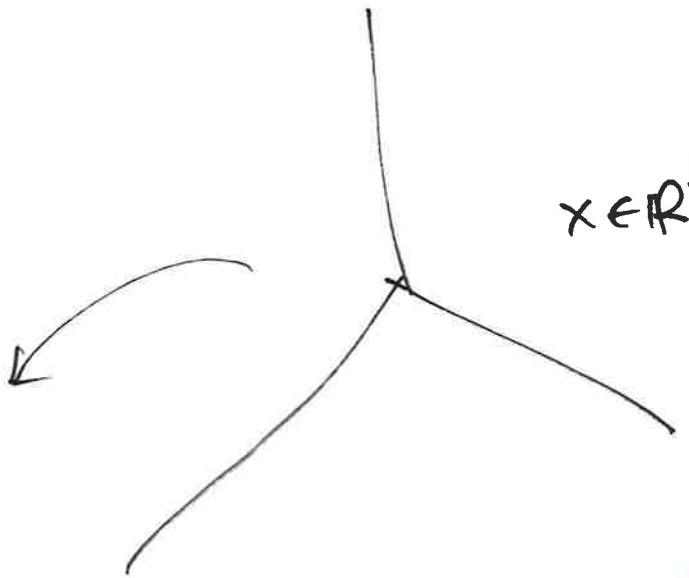
$$Ax = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3x_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \underbrace{(x_1 + 2x_2 + 3x_3)}_{\alpha} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$= \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \alpha \in \mathbb{R}$$





2-7

- the "right null space" (null space) of A is the set of all vectors x such that $Ax = 0$,

$$N = \{ x \text{ such that } Ax = 0 \}.$$

(also known as the "kernel")

- one interpretation of $Ax = 0$ is that

$$x_1 \begin{matrix} | \\ a_1 \\ | \end{matrix} + \cdots + x_n \begin{matrix} | \\ a_n \\ | \end{matrix} = 0$$

- if the columns of A are linearly independent, then the right null space of A contains only the vector $x = 0$.

- example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$Ax = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \rightarrow x = 0$$

$$\rightarrow N = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

- example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow N = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \in \mathbb{R}^2$ $(x \in \mathbb{R}^2)$.

$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow N = \{ 0 \} \in \mathbb{R}$ $(x \in \mathbb{R})$

- example: $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$

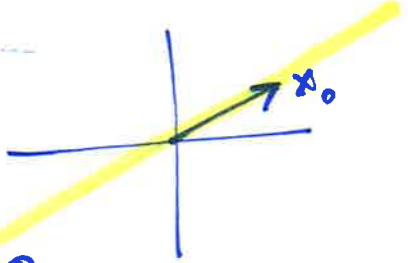
$x = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \rightarrow Ax = -3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$x = \begin{bmatrix} -3\alpha \\ \alpha \end{bmatrix} \rightarrow Ax = -3\alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$N = \left\{ \alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$

---

aside: $\boxed{x_0 \in N \rightarrow \alpha x_0 \in N}$



$Ax_0 = 0 \rightarrow A(\alpha x_0) = \alpha \underbrace{(Ax_0)}_{0} = \alpha \cdot 0 = 0$

- another interpretation of $Ax = 0$ is that x is orthogonal to all rows of A

$$\begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \quad a_i^T x = 0 \quad \forall i$$

- example: $A = [1 \quad -1] \rightarrow Ax = \overbrace{[1 \quad -1]}^{a_1^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

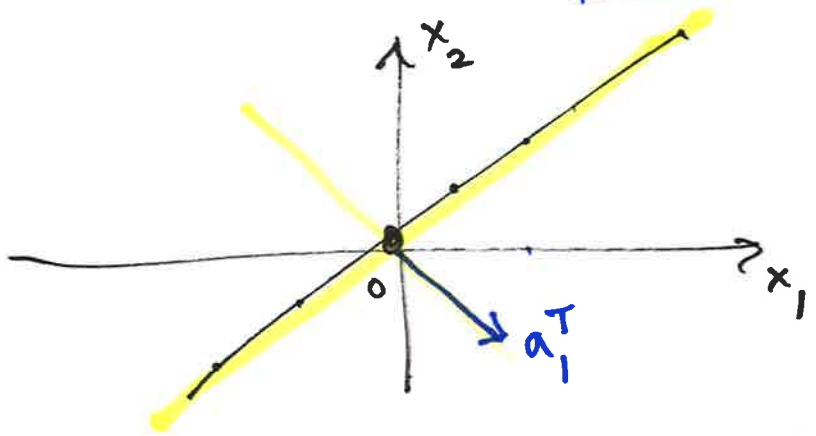
$$x_1 - x_2 = 0 \rightarrow x_2 = x_1$$

$$Ax = a_1^T x = 0$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \perp \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\downarrow$

$N = \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}\}$



- $y^T A$ gives a linear combination of the rows of A

$$y^T A = [y_1 \cdots y_m] \begin{bmatrix} - a_1^T - \\ \vdots \\ - a_m^T - \end{bmatrix} = y_1 a_1^T + \cdots + y_m a_m^T$$

- the "row space" of A is the set of all vectors z, $z^T = y^T A$, as y varies over all possible vectors in $\mathbb{R}^m$.

$$\mathcal{R} = \{ z \text{ such that } z^T = y^T A, \; y \in \mathbb{R}^m \}$$

- the "left null space" of A is the set of all vectors y such that $y^T A = 0^T$.

$$\mathcal{K} = \{ y \text{ such that } y^T A = 0^T \}$$

- if the rows of A are linearly independent, then the left null space of A contains only the vector $y^T = 0^T$.

$$\mathcal{K} = \{ 0 \}.$$

- it can be shown that the number of linearly independent columns of a matrix is equal to its number of linearly independent rows.

$$\text{rank}(A) = \text{number of linearly independent columns of } A$$

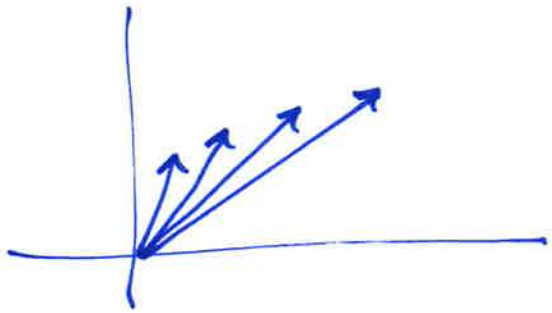
$$= \text{number of linearly independent rows of } A$$

- example: $A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$

# of lin. ind. col. of $A = 2 \longrightarrow \text{rank}(A) = 2$.

row vectors: $\begin{bmatrix} 1 \\ 5 \end{bmatrix}^T, \begin{bmatrix} 2 \\ 6 \end{bmatrix}^T, \begin{bmatrix} 3 \\ 7 \end{bmatrix}^T, \begin{bmatrix} 4 \\ 8 \end{bmatrix}^T$

# of lin. ind. rows of $A = 2$.



- example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$

$\text{rank}(A) = 1$

---

aside 1:

$\text{rank}(A) \leq \min\{m, n\}$ $\qquad A: m \times n$

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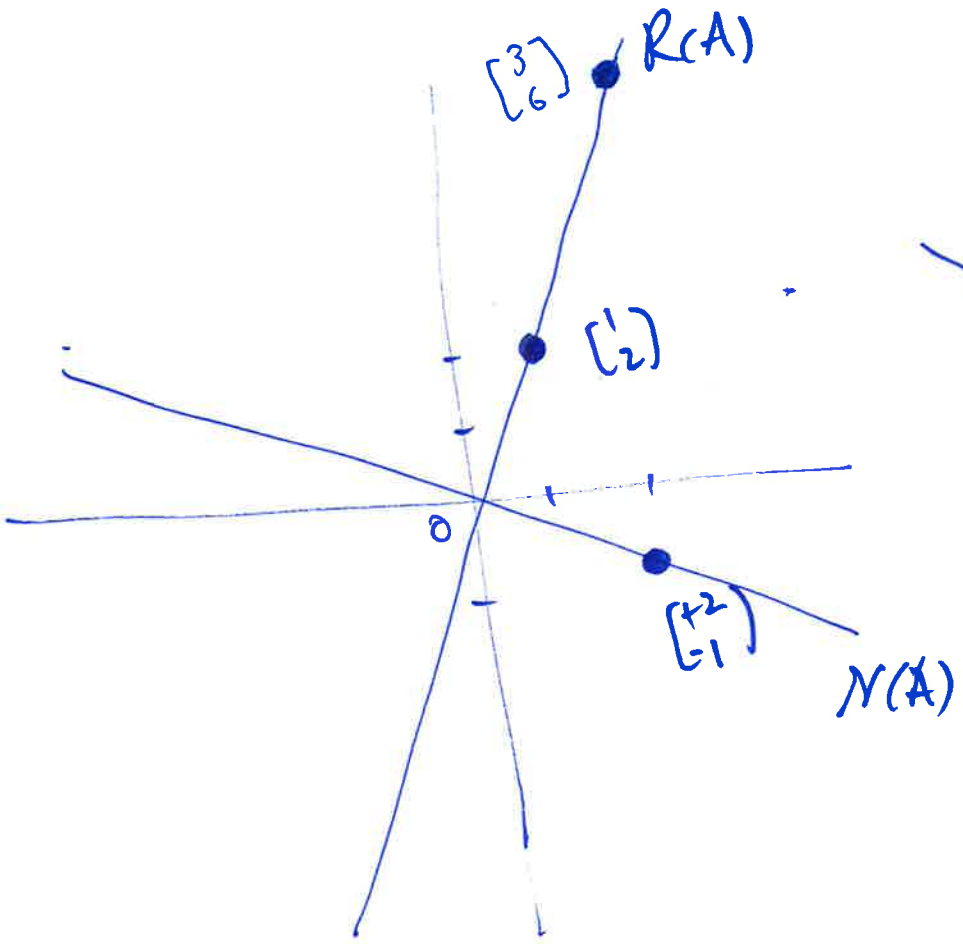
aside 2:

if A is $n \times n$, then

$\text{rank}(A) = n \iff \mathcal{C}(A) = \mathbb{R}^n \iff N(A) = \{0\}$

$\iff \mathcal{R}(A) = \mathbb{R}^n \iff \mathcal{N}(A) = \{0\}$

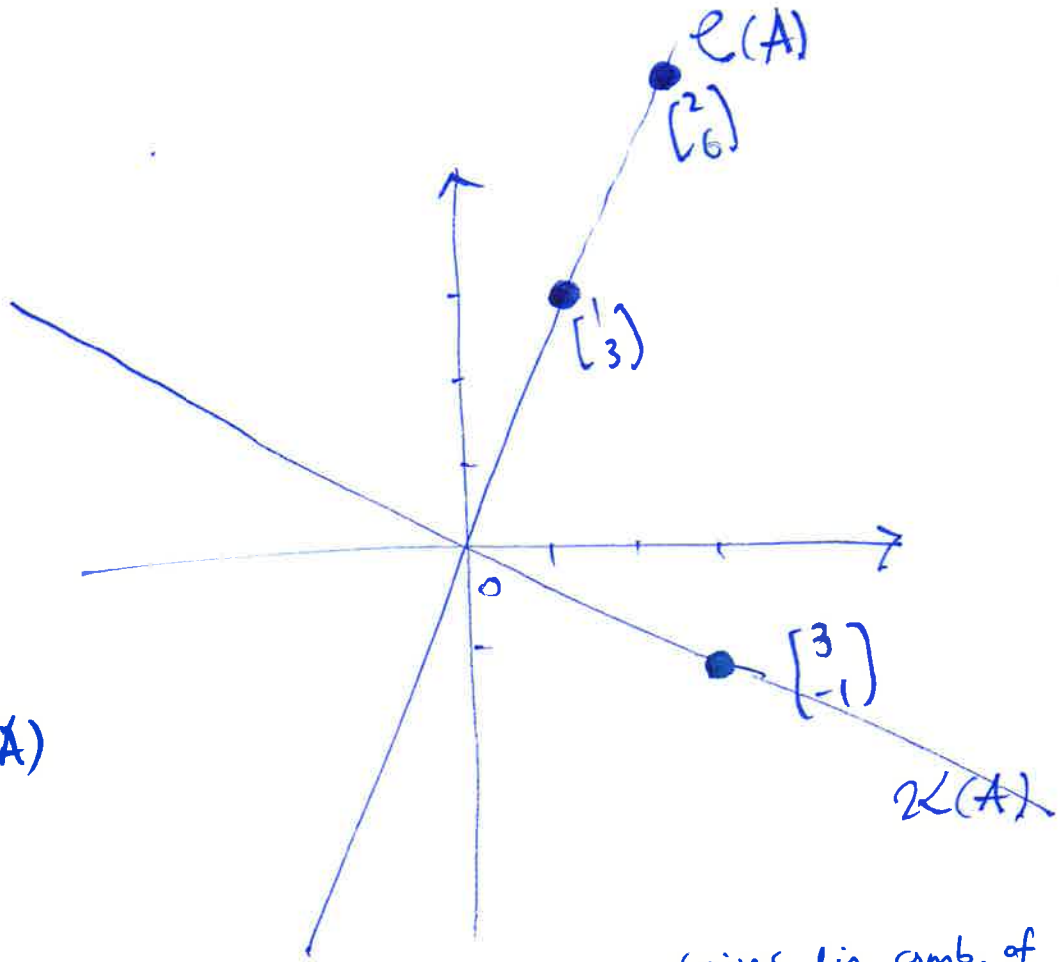
• example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$



$y^T A$ (gives lin. comb. of rows of A)

$Ax = 0$ (such x is orthog. to rows of A)

$\boxed{N(A) \perp R(A)}$

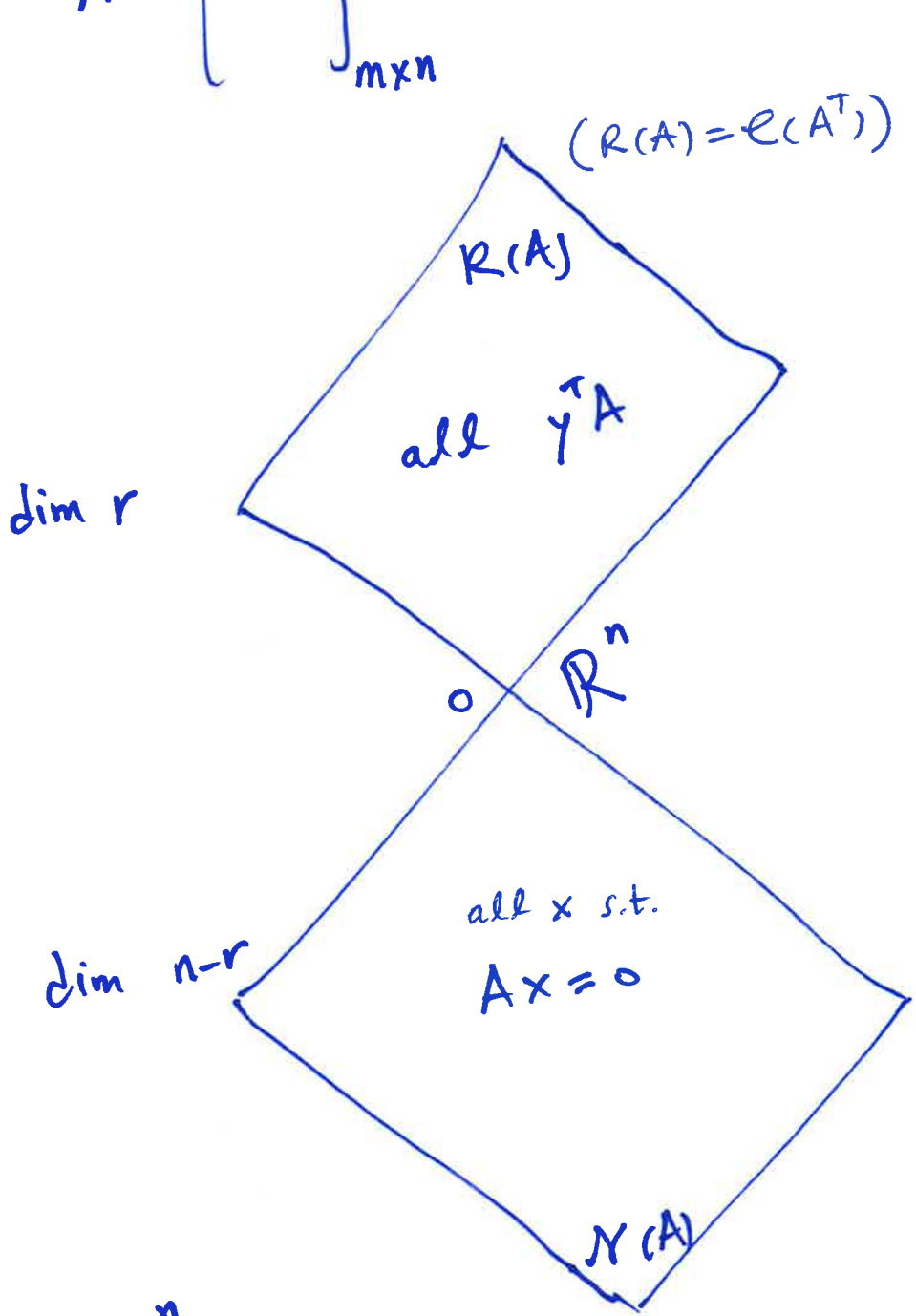


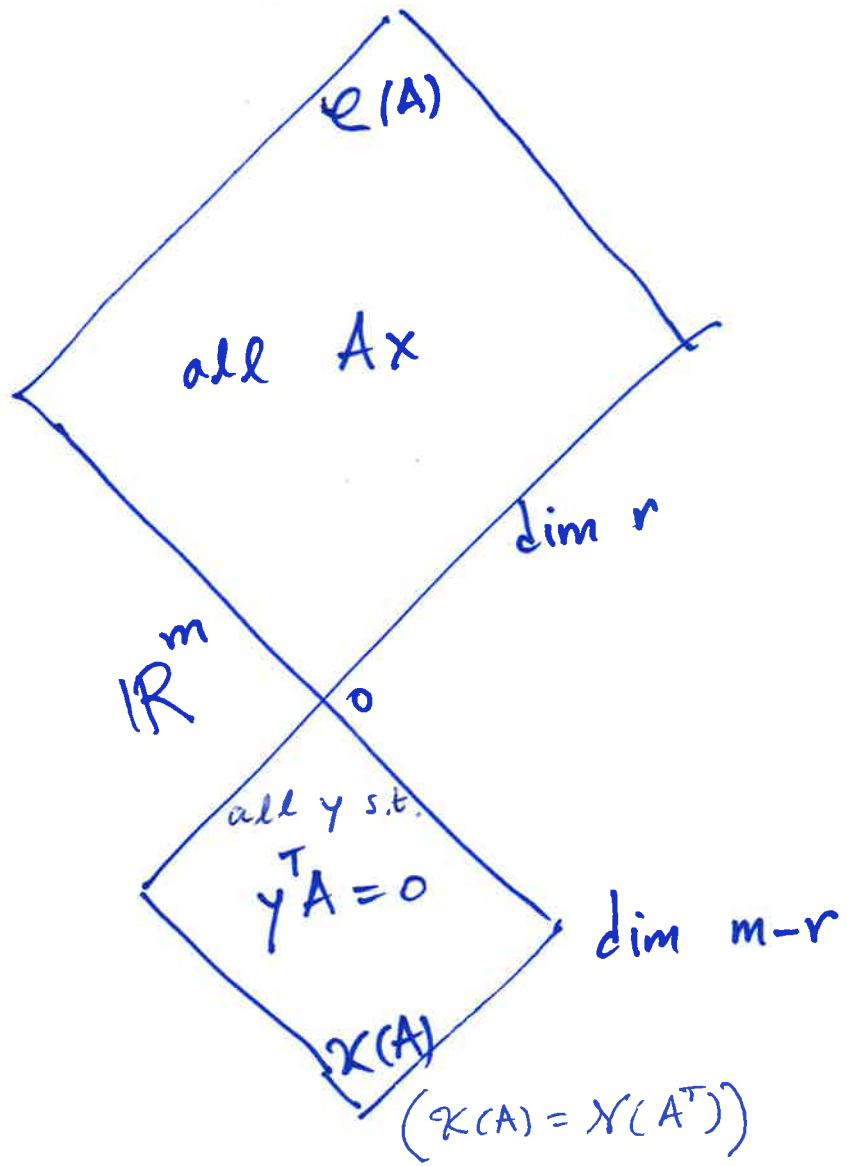
Ax (gives lin. comb. of columns of A)

$y^T A = 0$ (such y is orthog. to columns of A)

$\boxed{N(A) \perp C(A)}$.

$A = \begin{bmatrix} & \\ & \end{bmatrix}_{m \times n}$ & $\text{rank}(A) = r \leq \min\{m, n\}$.





$x \in \mathbb{R}^n \rightarrow x = x_R + x_N$

$y \in \mathbb{R}^m \rightarrow y = y_e + y_x$

- example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad m = n = 2, \quad r = 2$

$R(A) = \mathbb{R}^2$ $\qquad$ $\mathcal{C}(A) = \mathbb{R}^2$

$N(A) = \{0\}$ $\qquad$ $\mathcal{K}(A) = \{0\}$

- example: $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad m = 2, \; n = 1, \; r = 1$

$R(A) = \{\gamma\}$ $\qquad$ $\mathcal{C}(A) = \left\{\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$(\dim = r = 1)$ $\qquad$ $(\dim = r = 1)$

$N(A) = \{0\}$ $\qquad$ $\mathcal{K}(A) = \left\{\beta \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$

$(\dim = n - r = 1 - 1 = 0)$ $\qquad$ $(\dim = m - r = 2 - 1 = 1)$

- example: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad m = n = 2, \quad r = 2$

same as $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2-16

- example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ $m=2,\ n=3,\ r=1$

$R(A) = \left\{ \gamma \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ $(\dim = 1)$

$\mathcal{C}(A) = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ $(\dim = 1)$

$N(A) = \left\{ \delta \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ $(\dim = 2)$

$\mathcal{K}(A) = \left\{ \beta \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ $(\dim = 1)$

- some terminology $A: m \times n$, $\text{rank}(A) \leq \min\{m,n\}$
  - if $\text{rank}(A) = n$ we say A has "full column rank"
    (A is also said to be "one-to-one" or "injective")
  - if $\text{rank}(A) = m$ we say A has "full row rank"
    (A is also said to be "onto" or "surjective".)
  - if $\text{rank}(A) = \min\{m,n\}$ we say A has "full rank";
    otherwise it is "rank deficient" ($\text{rank}(A) < \min\{m,n\}$)

- if A has full row rank then $Ax = y$ has a solution for any y

$$\begin{cases} m \text{ lin. ind. rows} \Rightarrow m \text{ lin. ind. col.} \\ y \in \mathbb{R}^m \end{cases} \Rightarrow y \in \mathcal{C}(A) = \mathbb{R}^m$$

- consider $Ax = y$, x unknown, y & A known

we ask 2 questions:

① when does $Ax = y$ have a solution?

② assuming a solution exists, when is it unique?

answer to ①: $Ax = x_1 a_1 + \cdots + x_n a_n = y$

we have a solution if y belongs to the column space of A.

answer to ②: if the right null space of A contains only the vector zero (col. of A are lin. ind.) the solution is unique.

proof by contradiction: suppose sol. not unique

$$\begin{cases} Ax = y \\ A\tilde{x} = y \end{cases} \rightarrow Ax = A\tilde{x} \rightarrow A(x - \tilde{x}) = 0 \longrightarrow Az = 0$$

some lin. comb. of col. of A is zero, which contradicts assumption of lin. ind.

we can show that independence of columns of A is also necessary.

proof by contradiction: suppose col. not ind.

$$\begin{cases} Az = 0 \\ Ax = y \end{cases} \rightarrow A(x+z) = y$$

$$\downarrow$$

$$A\tilde{x} = y$$

which cotradicts uniqueness.

- example: $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$. $Ax = y$

  - a solution always exists (for any y) because the columns of A "span" the whole space ($\mathcal{C}(A) = \mathbb{R}^2$, $y \in \mathbb{R}^2$).
  - the solution is unique because the columns of A are linearly independent ($N(A) = \{0\}$, $x \in \mathbb{R}^2$).

- example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ $Ax = y$

  - a solution always exists because $\mathcal{C} = \mathbb{R}^2$, $y \in \mathbb{R}^2$.
  - the solution is not unique because the col. of A are lin. dependent ($a_1 = a_2 + a_3 \rightarrow a_1 - a_2 - a_3 = 0$)

$y = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, solve $Ax = y \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $Ax_0 = y$.

$x_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $Ax_1 = y$.

---

there exist many more solutions (infinitely many of them); how do we characterize them all?

$z = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \rightarrow Az = 0 \rightarrow N = \left\{ \alpha \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$.

$x_0 + \alpha z = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ 1+\alpha \\ \alpha \end{bmatrix}$ is a solution of $Ax = y$ for <u>any</u> value of $\alpha \in \mathbb{R}$.

$A(x_0 + \alpha z) = Ax_0 + A(\alpha z)$

$= Ax_0 + \alpha \underbrace{Az}_{0} = Ax_0 = y$.

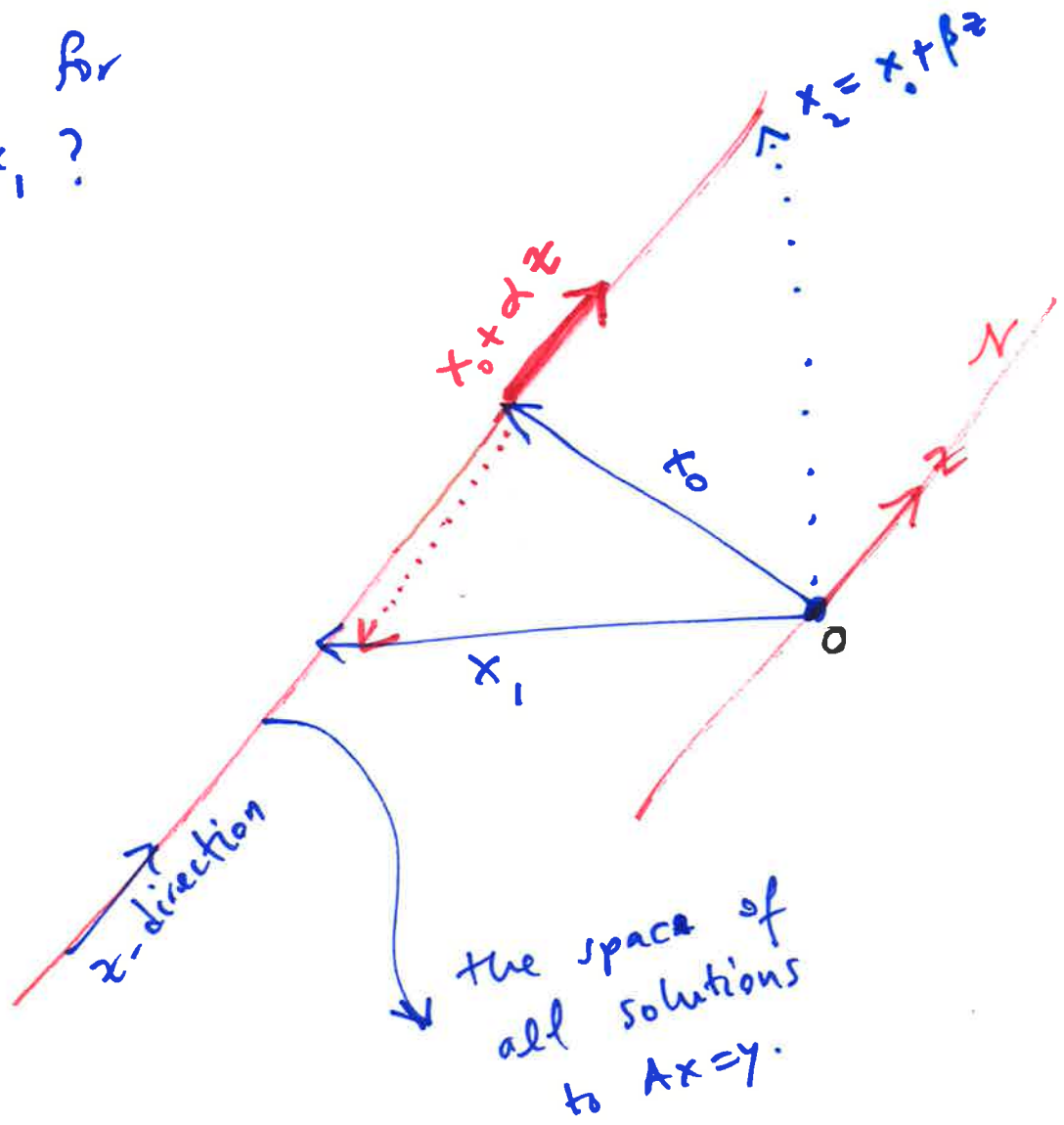
question: is there some α for which $x_0 + \alpha z = x_1$?

answer: yes, try $\alpha = 1$

$$x_0 + 1z = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= x_1 \quad \checkmark$$



---

aside:

this has strong resemblance to finding solutions to linear differential equations:

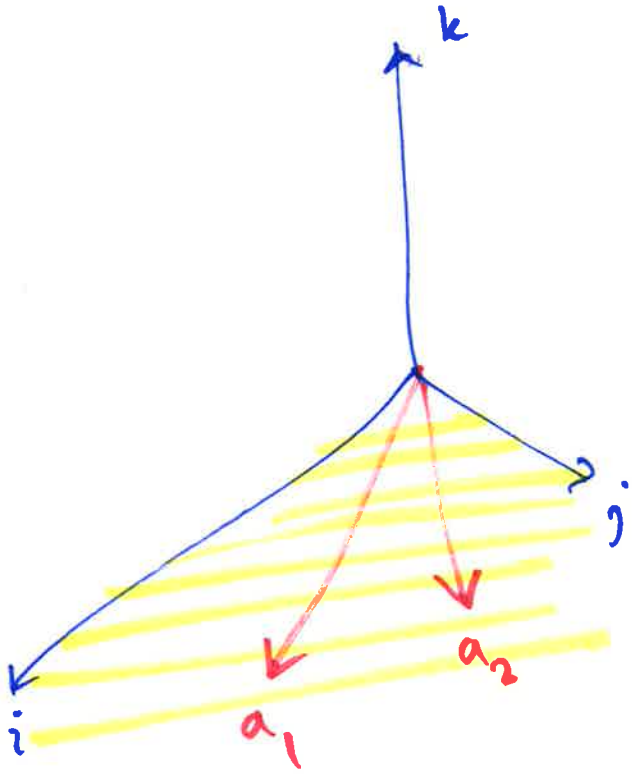
$$\mathcal{D}[x(t)] = y(t)$$

$\mathcal{D}$: linear differential operator

$$\mathcal{D}\left[\underbrace{x_p(t)}_{\text{particular}} + \underbrace{\sum_{i=1}^{q} \alpha_i x_h(t)}_{\text{spans space of}}\right] = y(t)$$

- example: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad Ax = y \quad$ $x \in \mathbb{R}^2$, $y \in \mathbb{R}^3$

$y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  $\qquad y = \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$ 



$\mathcal{C} \neq \mathbb{R}^3$ ($y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solution)

but, if $y = \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$, $\alpha, \beta \in \mathbb{R}$,

then there is a solution (because $y \in \mathcal{C}$)

and in this case the solution is unique.

- if A is $n \times n$ and $\text{rank}(A) = n$

↓

$\mathcal{C}(A) = \mathbb{R}^n$

$N(A) = \{0\}$

↓

$Ax = y$ always has unique solution

↓

A^{-1} exists & $x = A^{-1}y$ is the unique solution for any $y \in \mathbb{R}^n$.

(this creates a one-to-one correspondence between y's & x's.)

---

aside:

$$Ax = y \rightarrow A^{-1}(Ax) = A^{-1}y \rightarrow Ix = A^{-1}y \rightarrow x = A^{-1}y.$$

I

- question: what if $y \notin \mathcal{C}(A)$?

$Ax = y$ has no solution

if $Ax = y \rightarrow Ax - y = 0 \rightarrow \|Ax - y\| = 0$

& if $Ax \neq y \rightarrow Ax - y \neq 0 \rightarrow \|Ax - y\| \neq 0$

think of $e := Ax - y$ as error or "mismatch" between Ax & y. the best we can do is choose x such that $\|e\| = \|Ax - y\|$ is as small as possible.

↓

least-squares

$\|e\|_2^2 = e_1^2 + \cdots + e_n^2$ is as small as possible by choosing a good x.

↓

$e^T e$

the least-squares problem

- suppose it is desired to find the vector x such that $e = Ax - y$ has minimum length

$$\underset{x}{\text{minimize}} \quad \|Ax-y\|_2^2$$

$$\left(\|e\|_2^2 := \sum_{i=1}^{n} e_i^2 \right)$$

assume that $N(A) = \{0\}$ ... (col. of A lin. ind.)

$$A \sim \left[\begin{array}{c} \text{tall} \end{array}\right]_{m \times n}, \quad m \geq n, \quad \text{rank}(A) = n.$$

A has full column rank

$$Ax = x_1 a_1 + \cdots + x_n a_n = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} x$$

$A \in \mathbb{R}^{m \times n}$

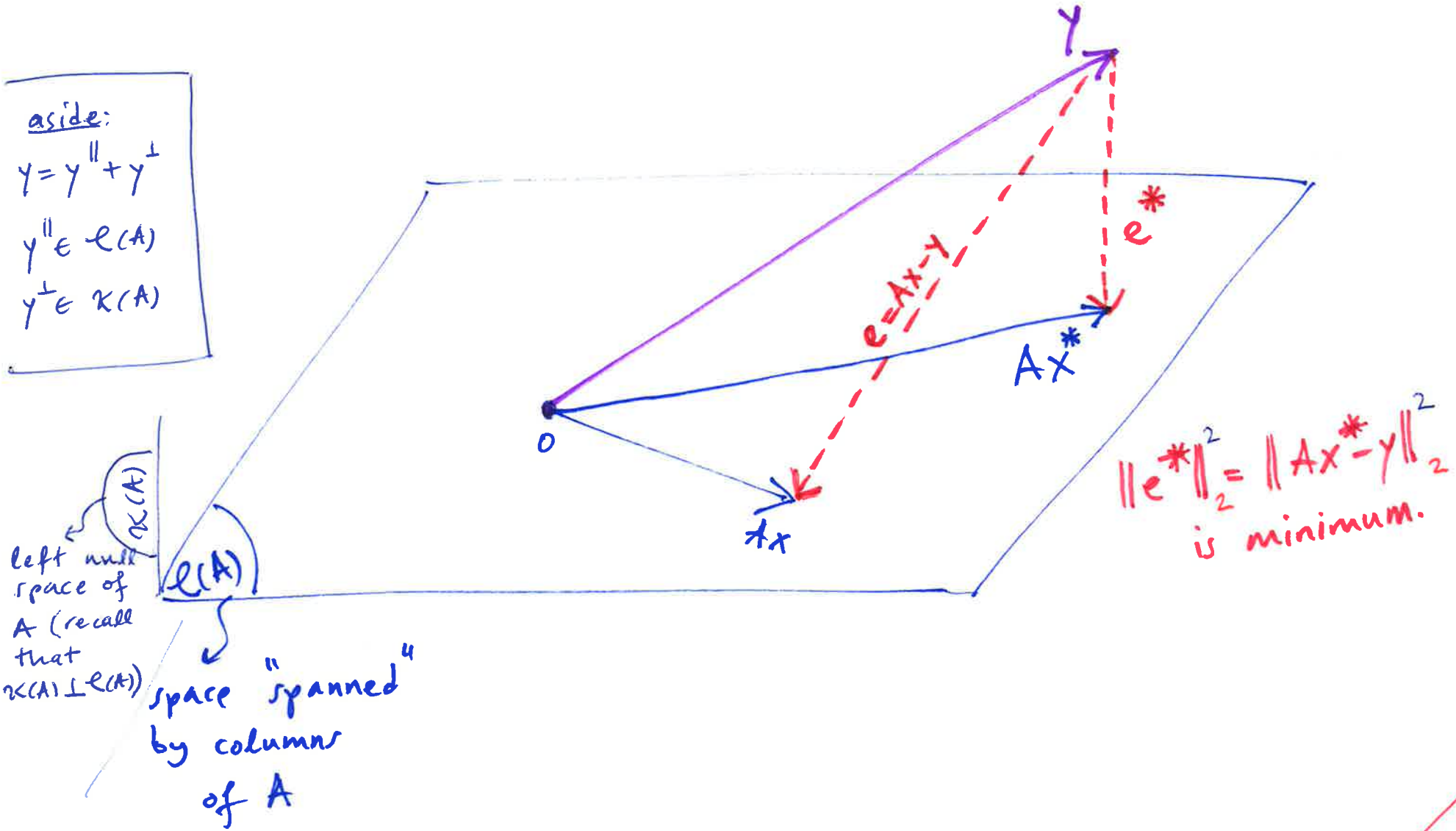
$y \in \mathbb{R}^m$

$x \in \mathbb{R}^n$

$e \in \mathbb{R}^m$

$Ax \in \mathbb{R}^m$

$$e = Ax - y$$



for e^* to have minimum length, we need it to be "orthogonal to all vectors in the column space of A"

$\Updownarrow$

$e^* \perp a_1, \ldots, e^* \perp a_n$ ($\{a_i, i=1,\ldots,n\}$ form basis for $C(A)$)

$(e^*)^T a_1 = 0, \ldots, (e^*)^T a_n = 0$

$$(e^*)^T \begin{bmatrix} | & | & | \\ a_1 & \cdots & a_n \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} = 0^T \qquad (e^* \in N(A^T))$$

$(e^*)^T A = 0^T$

$A^T e^* = 0$

$A^T (A x^* - y) = 0$

$A^T A x^* - A^T y = 0$

$A^T A x^* = A^T y \longrightarrow$ $\boxed{x^* = (A^T A)^{-1} A^T y}$ (see comment)

aside:

$(A \pm B)^T = A^T \pm B^T$

$(AB)^T = B^T A^T$

$(A^T)^T = A$

2-27

- comment regarding invertibility of A^TA:
  - A^TA is square (as is AA^T)
  - if $N(A) = \{0\}$ (col's of A are lin. ind.)

    then (HW problem) A^TA is invertible

---

* where is x in picture?

- example: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} 29 \\ 30 \\ 31 \end{bmatrix}$, minimize $\|Ax-y\|_2$ over x

  A $\quad$ y

$Ax = y$ has no solution ($\mathcal{C}(A) = \left\{\begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}\right\}$, and $y = \begin{bmatrix} 29 \\ 30 \\ 31 \end{bmatrix}$ is not in $\mathcal{C}(A)$)

$A^TAx^* = A^Ty$. gives the least-squares solution

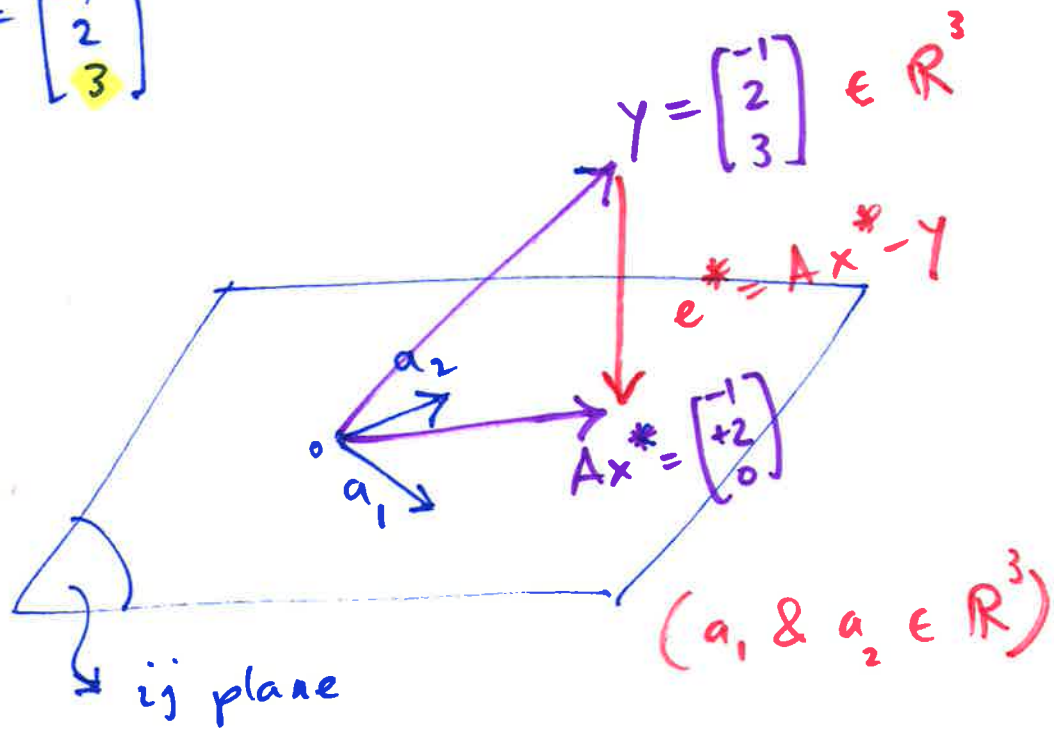
$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x^* = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 29 \\ 30 \\ 31 \end{bmatrix}$$

$3x^* = 29+30+31 \longrightarrow x^* = \dfrac{29+30+31}{3}$ = average ! ✓

2-28

- example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

a_1 a_2



by inspection:

$$Ax^* = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

$$x^* = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

least-squares solution:

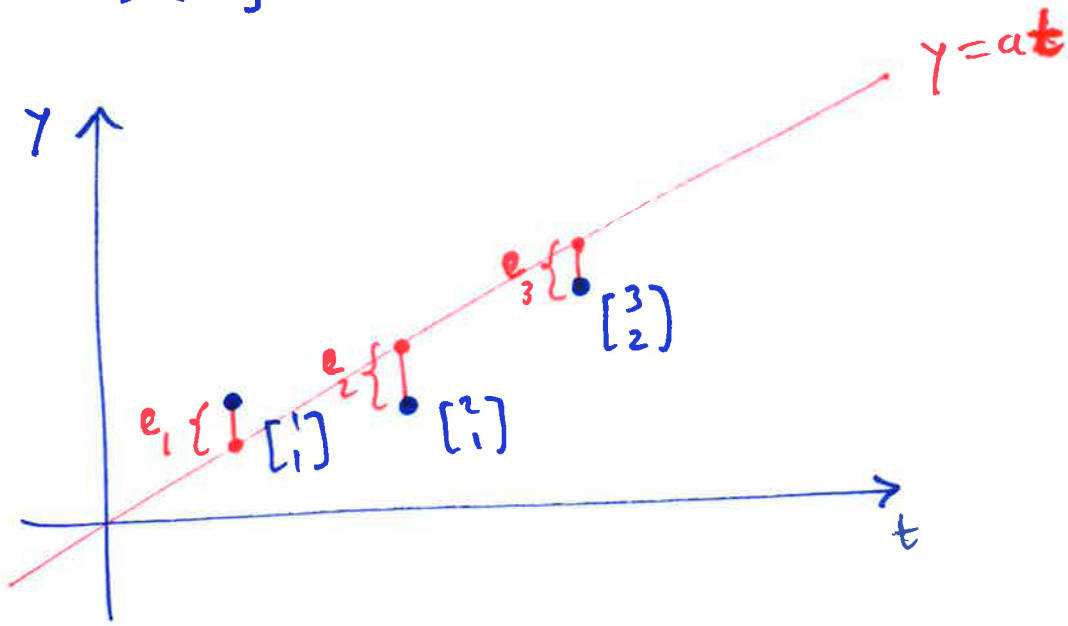
$$A^T A x^* = A^T y$$

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^T y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A^T A x^* = A^T y \longrightarrow x^* = A^T y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \checkmark$$

$A^T A$ → I

• example: find "a" such that the line $y = at$ minimizes $e_1^2 + e_2^2 + e_3^2$



method 1

$$J(a) = e_1^2 + e_2^2 + e_3^2$$

$$= \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ a.1 \end{bmatrix} \right\|^2$$

$$+ \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ a.2 \end{bmatrix} \right\|^2$$

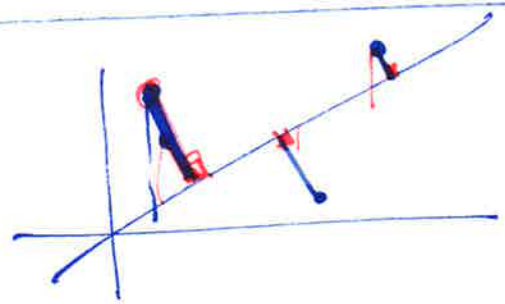
$$+ \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ a.3 \end{bmatrix} \right\|^2 = (1-a)^2 + (1-2a)^2 + (2-3a)^2$$

$$\frac{\partial J}{\partial a} = 0 \rightarrow 28\,a^* - 18 = 0 \rightarrow a^* = \frac{9}{14}.$$

---

aside:

$$|e_1| + |e_2| + |e_3|. \quad \text{or}$$



method 2

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} a - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$A \quad x \quad y$

minimize $\|e\|^2 = \|Ax - y\|^2$.

$$a^* = (A^T A)^{-1} A^T y$$

$$= \left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{9}{14}.$$

---

aside: suppose A were fat

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

$$A^T A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

$\text{rank}(A^T A) = 1$

$$(A^T A)^{-1} \sim \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \frac{1}{4-4}$$

- when A satisfies $N(A) = \{0\}$ (A has lin. ind. col's, then the matrix & thus A^TA is invertible)

$$A^\dagger := (A^TA)^{-1}A^T$$

is a pseudoinverse of A (also referred to as the left inverse of A)

$$A^\dagger A = (A^TA)^{-1}A^TA \qquad (AA^\dagger \text{ may or may not be } I)$$

$$= I$$

- simple example: $A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$A^\dagger = \left(\begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}$$

$$A^\dagger A = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 1$$

$$AA^\dagger = \begin{bmatrix} 2 \\ 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

the least-norm problem

- there is a "dual" (mirror image) to the least-squares problem, which is also in the form of an optimization problem

- suppose it is desired to find the vector x of minimum length such that $Ax = y$

$$\begin{aligned} &\text{minimize} && \|x\|_2^2 \\ &\text{subject to} && Ax = y. \end{aligned}$$

assume that $\mathcal{K}(A) = \{0\}$ (rows of A are lin. ind.)

$$A \sim \underbrace{\left[\ \text{fat}\ \right]}_{m \times n}, \quad m \leq n, \quad \text{rank}(A) = m$$

A has full row rank

---

aside:

① $\mathcal{K}(A) = \{0\} \rightarrow$ m lin. ind. rows $\rightarrow$ (at least) m lin. ind. col. ($\in \mathbb{R}^m$)

② $y \in \mathbb{R}^m$

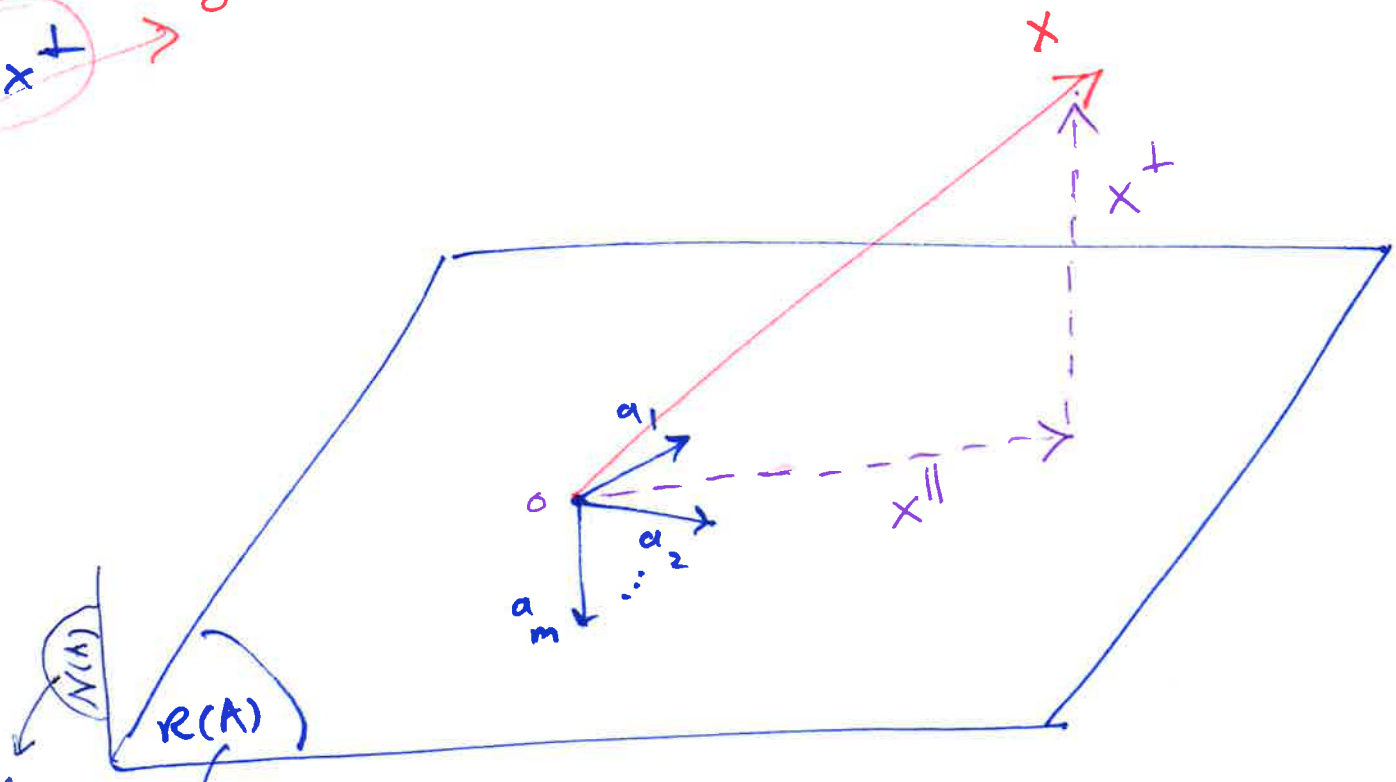
} ① & ② $\downarrow$ $Ax = y$ always has a solution

$$Ax = y \rightarrow \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$Ax = A(x^{\parallel} + x^{\perp})$$

$$= Ax^{\parallel} + Ax^{\perp} \rightarrow 0$$

$$= y$$



aside:
$x = x^{\parallel} + x^{\perp}$
$x^{\parallel} \in R(A)$
$x^{\perp} \in N(A)$

right null space of A
(recall that $N(A) \perp R(A)$)

row space of A

- $Ax^{\perp}$ is always zero because $x^{\perp}$ is by definition orthogonal to the rows of A. therefore the component $x^{\perp}$ does not affect $Ax=y$ and only adds to the length of x.

- therefore the x^{*} with the minimum norm should be chosen inside the row space of A (i.e., $x^{\perp}=0$)

$$x^{*} = z_1^{*} a_1 + \cdots + z_m^{*} a_m = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix} \begin{bmatrix} z_1^{*} \\ \vdots \\ z_m^{*} \end{bmatrix} = A^{T} z^{*}$$

in summary $\begin{cases} Ax^{*} = y \\ x^{*} = A^{T} z^{*} \end{cases} \rightarrow y = Ax^{*} = A(A^{T} z^{*})$

$y = AA^T z^*$ → $z^* = (AA^T)^{-1} y$

always invertible if A has linearly independent rows. ($\mathcal{N}(A) = \{0\}$)

↓

$x^* = A^T z^*$

↓

$$\boxed{x^* = A^T (AA^T)^{-1} y}$$

compare with least-squares solution

$x^* = (A^T A)^{-1} A^T y$.

---

aside: AA^T not invertible if A tall

$$A = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow AA^T = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}.$$

review of last lecture

- Ax gives linear combination of col's of A

$$Ax = x_1 a_1 + \cdots + x_n a_n \qquad A: m \times n \qquad A \sim \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$$

- col. space $\mathcal{C} = \{ Ax \text{ as } x \text{ varies over all vectors in } \mathbb{R}^n \}$

- right null space $N = \{ x \text{ such that } Ax = 0 \}$.

  - if columns of A are lin. ind. $\rightarrow N = \{0\}$

  - $Ax = 0 \rightarrow a_i^T x = 0 \quad \forall i$

    a_i^T is ith row of A

review cont'd

- $x^T A$ gives linear combination of rows of A

$$x^T A = x_1 a_1^T + \cdots + x_m a_m^T \qquad A: m \times n \qquad A \sim \begin{bmatrix} -a_1^T- \\ \vdots \\ -a_m^T- \end{bmatrix}$$

- row space $\mathcal{R} = \{ x^T A \text{ as } x \text{ varies over all vectors in } \mathbb{R}^m \}$

- left null space $\mathcal{K} = \{ x \text{ such that } x^T A = 0^T \}$

  - if rows of A are lin. ind $\rightarrow$ $\mathcal{K} = \{0\}$.

  - $x^T A = 0^T \rightarrow x^T a_i = 0 \quad \forall i$

    a_i is i^{th} column of A

- $\mathcal{C}, \mathcal{N}, \mathcal{R}, \mathcal{K}$ are all unbounded

  e.g. $y \in \mathcal{C} \rightarrow \alpha y \in \mathcal{C} \quad \forall \alpha \in \mathbb{R}$.

review of last lecture

- # of lin. ind. col. of A
  $=$ # of lin. ind. rows of A
  $=:$ rank (A) $\qquad (\leq \min\{m,n\})$

(implications for $n \times n$ matrices with rank $= n$)

- consider solving $Ax = y$
  - solvability: $y \in \mathcal{C}(A)$
  - uniqueness: $N(A) = \{0\}$

- when $y \in \mathcal{C}(A)$, $N(A) \neq \{0\}$.
  - if x_p is a "particular" solution: $Ax_p = y$
  - if $x_h^{(i)}$ is a "homogeneous" solution: $Ax_h^{(i)} = 0$, $i = 1, \cdots, r$

  $\longrightarrow$ $x_p + \sum_{i=1}^{r} \alpha_i x_h^{(i)}$ is also a solution for any $\{\alpha_i\}$

review of last lecture

- considered $Ax = y$
  - solvability $y \in \mathcal{C}(A)$
  - uniqueness $N(A) = \{0\}$
- what if (A is tall) and $y \notin \mathcal{C}(A)$ so that $Ax = y$ has no solution?

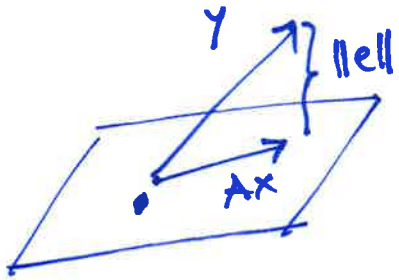
$A \sim$ [tall]

($Ax = y$ overdetermined system of equ.)

↓

then minimize $\|Ax - y\|$

i.e., find x^* such that $\|Ax^* - y\|$ is smallest among all possible $\|Ax - y\|$.



↓

the optimal x^* should be such that $e^* := Ax^* - y$ is orthogonal to col. space of A.

↓

$$A^T A x^* = A^T y \rightarrow \quad x^* = (A^T A)^{-1} A^T y.$$

review of last lecture

- what if (A is fat) and $Ax=y$ has many solutions?

  $N(A) \neq \{0\}$

  $A \sim [\ \text{fat}\ ]$

  ($Ax=y$ underdetermined system of equations)

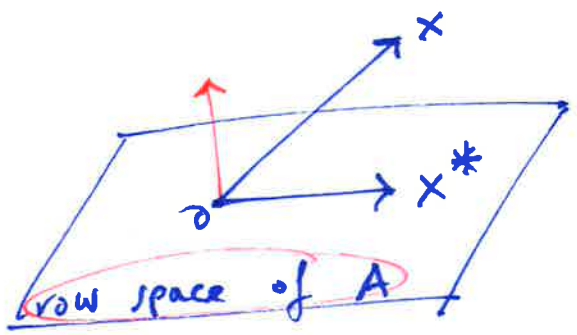
  ↓

  then find x^* such that $\|x^*\|$ is smallest among all x that satisfy $Ax=y$.

  minimize $\|x\|_2$
  subject to $Ax=y$.

  ↓

  the optimal solution x^* should belong to the row space of A

  ↓

$$\begin{cases} x^* = A^T z \\ Ax^* = y \end{cases} \rightarrow x^* = A^T(AA^T)^{-1}y.$$

- example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

only increases length of x without affecting y.

by inspection

$$x^* = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$x^\perp = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}, \quad a_1^T x^\perp = 0, \quad a_2^T x^\perp = 0.$$

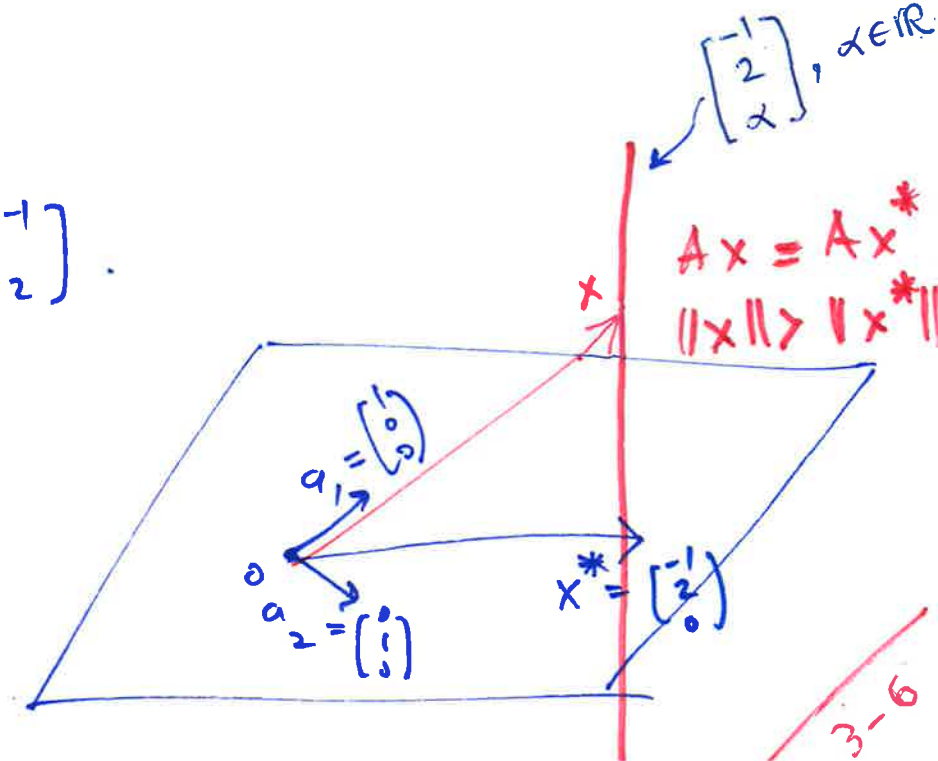
least-norm solution

$$x^* = A^T(AA^T)^{-1}y$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Big(\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{I} \Big)^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$



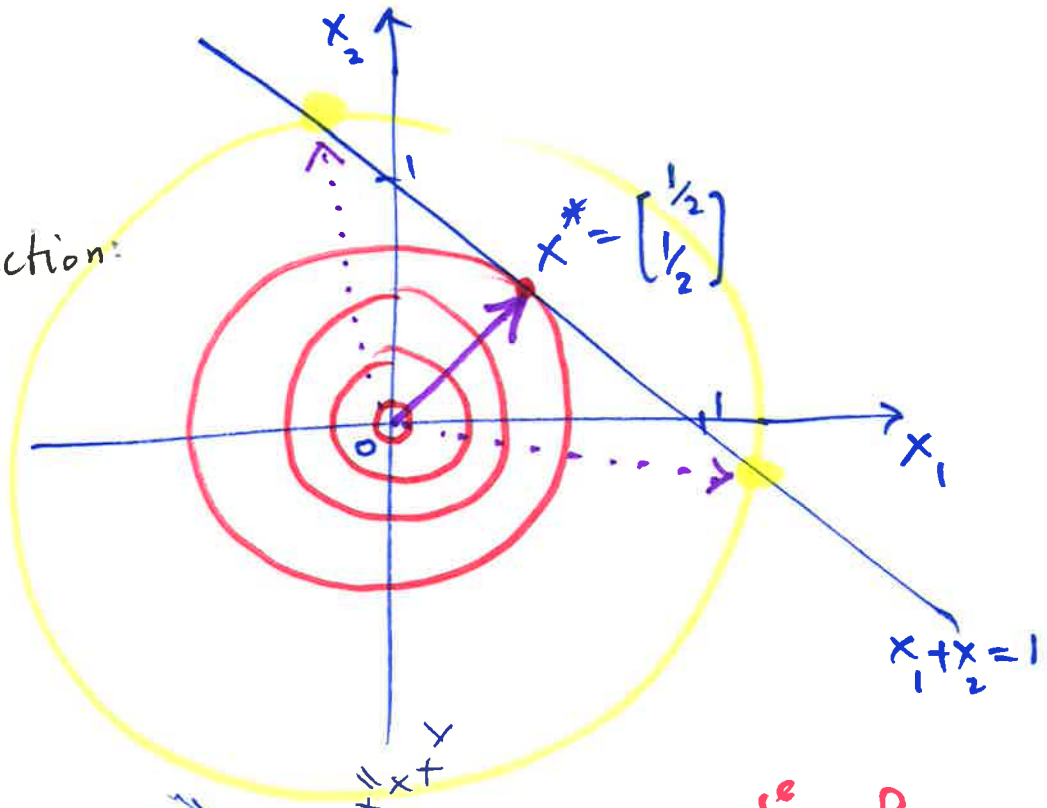
- example: $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$

$\updownarrow$

$x_1 + x_2 = 1$

find x with minimum length

by inspection:



$A = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$a_1^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x^{\parallel} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$, $x^{\perp} = \begin{bmatrix} \beta \\ -\beta \end{bmatrix}$

$(Ax^{\perp} = 0)$

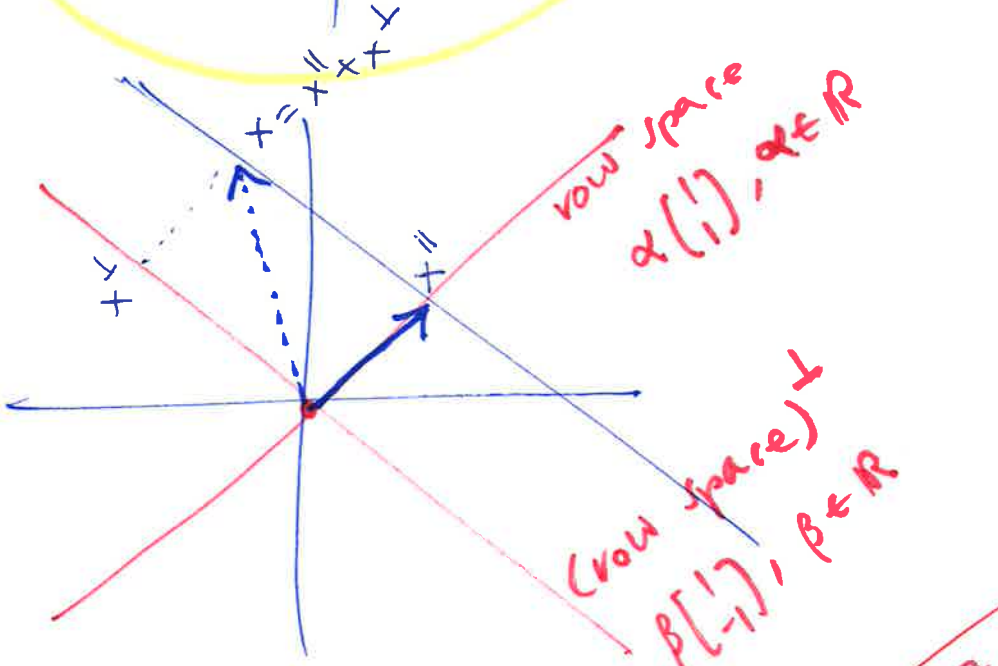
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using least-norm:

$x^* = A^T (AA^T)^{-1} y$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \cdot 1$

$= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



- when A satisfies $\mathcal{N}(A) = \{0\}$ (i.e., A has lin. ind. rows & thus AA^T is invertible)

$$A^+ := A^T(AA^T)^{-1}$$

is a pseudoinverse of A.

(also referred to as the right inverse of A)

$$AA^+ = AA^T(AA^T)^{-1} \qquad (A^+A \text{ may or may not be } I)$$
$$= I$$

- simple example: $A = \begin{bmatrix} 2 & 0 \end{bmatrix} \rightarrow A^+ = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$.

$$AA^+ = \begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = 1$$

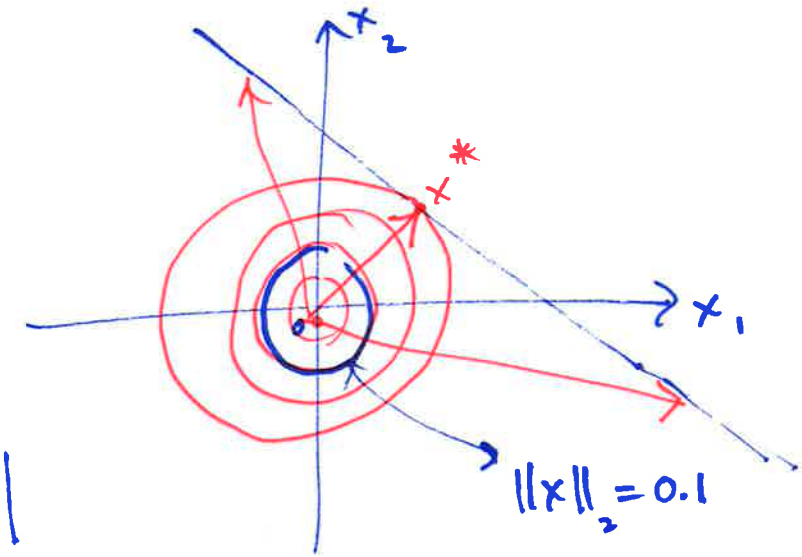
$$A^+A = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A function $f : \mathbf{R}^n \to \mathbf{R}$ with $\mathbf{dom}\, f = \mathbf{R}^n$ is called a *norm* if

- f is nonnegative: $f(x) \geq 0$ for all $x \in \mathbf{R}^n$
- f is definite: $f(x) = 0$ only if $x = 0$
- f is homogeneous: $f(tx) = |t|f(x)$, for all $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$
- f satisfies the triangle inequality: $f(x + y) \leq f(x) + f(y)$, for all $x, y \in \mathbf{R}^n$

$$\min. \quad \|x\|_2 := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$s.t. \quad x_1 + x_2 = 1 \rightarrow Ax = y$$

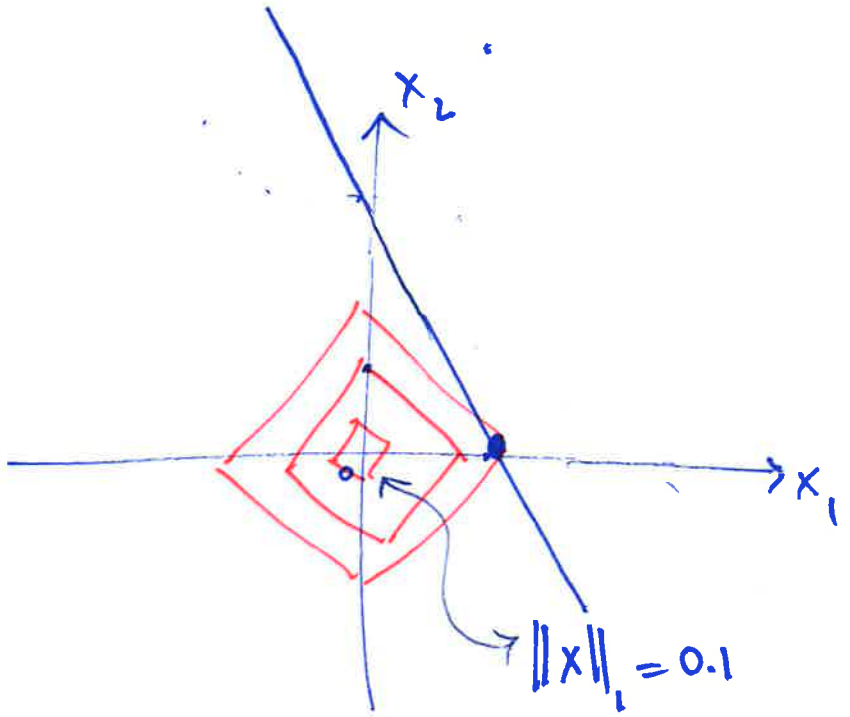


$$\min. \quad \|x\|_1 := |x_1| + |x_2| + \cdots + |x_n|$$

$$s.t. \quad Ax = y$$

$$\min. \quad \|x\|_1$$

$$s.t. \quad 2x_1 + x_2 = 1$$



$$\|x\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$

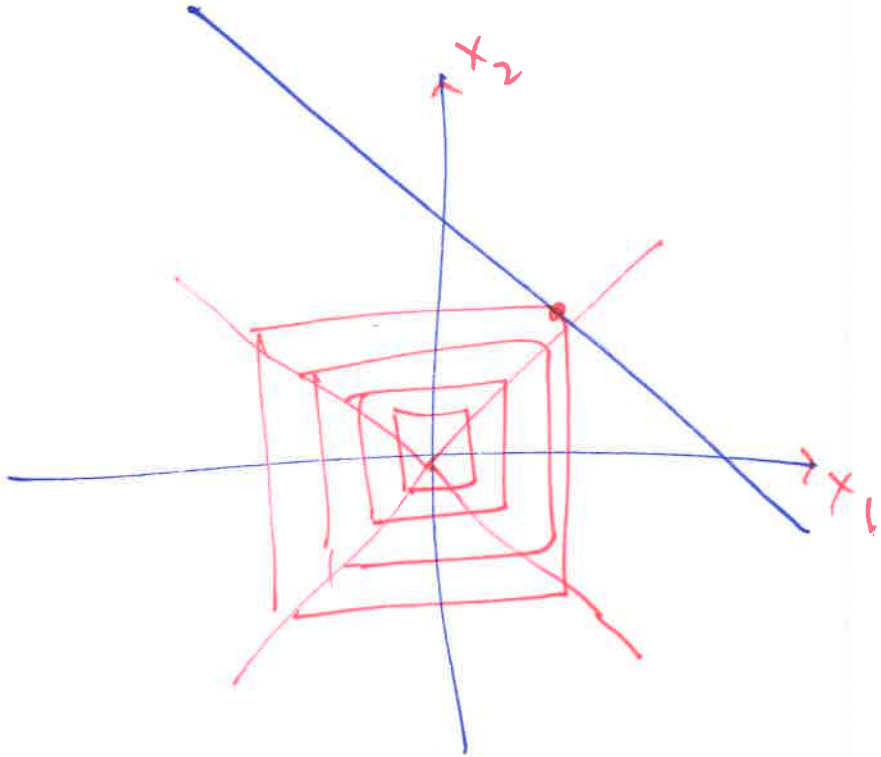
$$= \left(|x_1|^p + \cdots + |x_n|^p \right)^{1/p}$$

$p \to \infty$

$$= \max_i \{ |x_i| \}$$

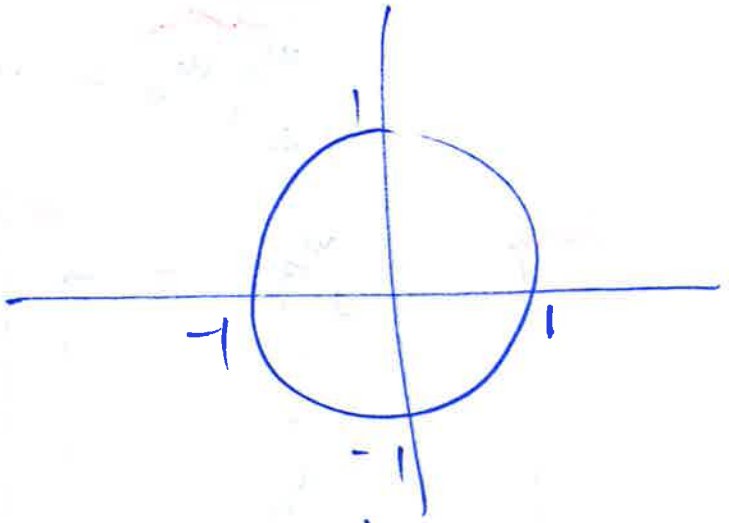
$$\|x\|_\infty = \max_i \{ |x_i| \}.$$

$$x = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad x = \begin{pmatrix} -0.5 \\ -0.7 \end{pmatrix}$$

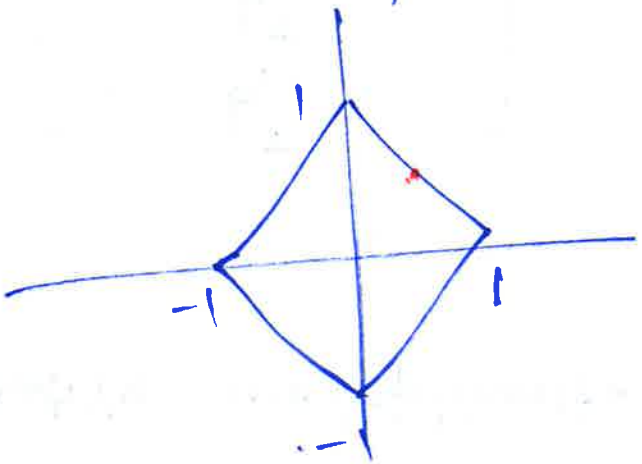


$$\|x\|_2 = 1$$

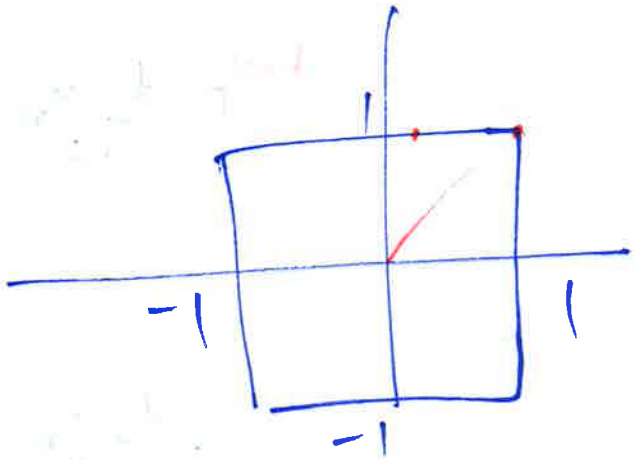
$$\underbrace{x^T x = 1}$$



$$\|x\|_1 = 1$$



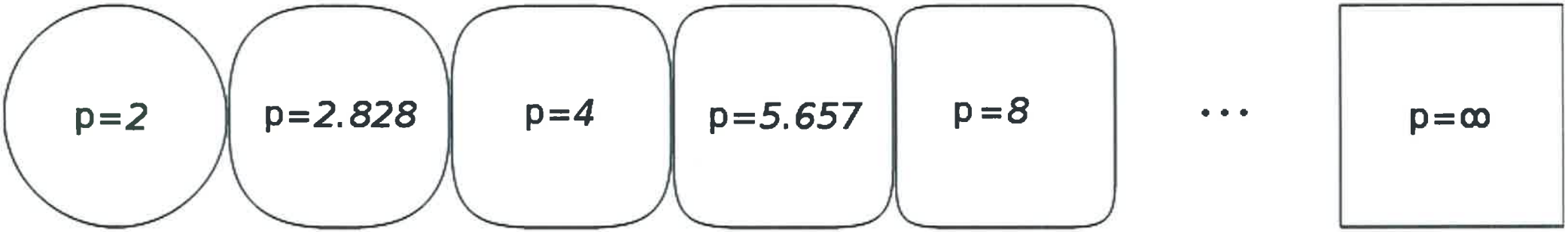
$$\|x\|_\infty = 1$$



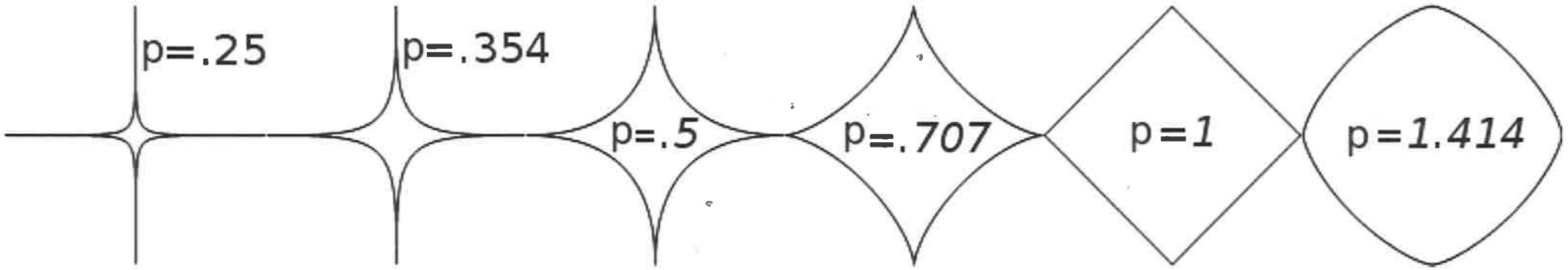
$$\|x\|_p = 1$$

p-norm of vector x:

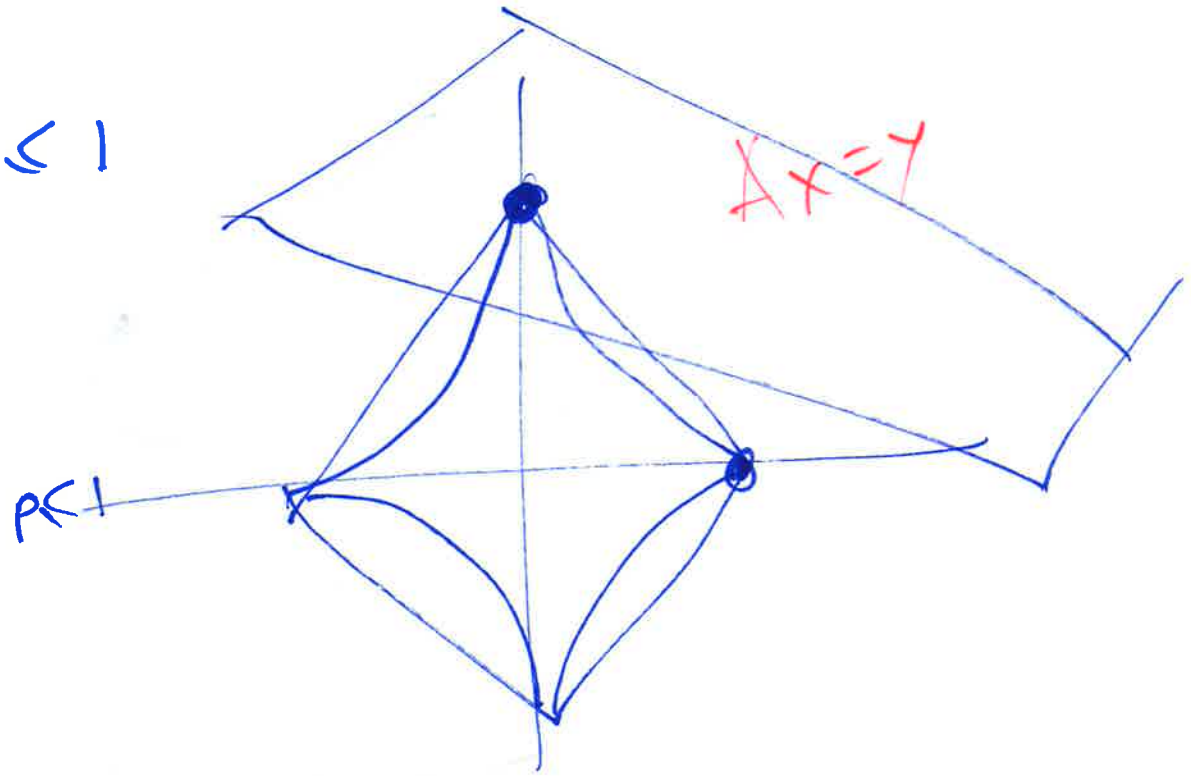
$$\|x\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}, \qquad 0 < p \leq \infty$$



$$\|x\|_\infty = \max_i |x_i|$$


p=.25
p=.354
p=.5
p=.707
p=1
p=1.414

$\|x\|_p \quad p \leq 1$



$\min. \|x\|_p$

$s.t. \quad Ax = y$

$$\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right)$$

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

→ sparse optimization & compressed sensing.

vector space & subspace

- a vector space over a "field" F (commonly, scalars in $\mathbb{R}$) is a "set" V (commonly, vectors in $\mathbb{R}^n$) with the operations of vector addition ($+ : V \times V \to V$) and scalar multiplication ($\cdot : F \times V \to V$) that satisfies the following:

①	$u + (v+w) = (u+v) + w$	for all $u, v, w \in V$	associativity of add.
②	$u + v = v + u$	for all $u, v \in V$	commutativity of add
③	$\exists\, \emptyset \in V$ s.t. $v + \emptyset = v$	for all $v \in V$	identity element of add.
④	$\forall v \in V$, $\exists -v \in V$ s.t. $v + (-v) = \emptyset$		inverse element of add.
⑤	$\alpha(\beta v) = (\alpha\beta) v$	for all $v \in V$; $\alpha, \beta \in F$.	compatibility of scalar mult. w/ field mult.
⑥	$1 \cdot v = v$	1 being mult. identity in F	identity of scalar mult.
⑦	$\alpha(u+v) = \alpha u + \alpha v$	for all $u, v \in V$; $\alpha \in F$	distributivity of scalar mult. w.r.t vec. add.
⑧	$(\alpha+\beta) v = \alpha v + \beta v$	for all $v \in V$; $\alpha, \beta \in F$	distributivity of scalar mult. w.r.t. field add.

- note that in the above definition, we have assumed that V is "closed" under vector addition and scalar multiplication, i.e.,

  $v + w \in V$ for all $v, w \in V$

  $\alpha v \in V$ for all $v \in V$; $\alpha \in F$.

- a subspace (linear subspace) is a subset W of a vector space V that is closed under vector addition and scalar multiplication. a subspace is itself a vector space (it inherits the properties of vector spaces)

- the span of a set S of vectors in V is the smallest subspace of V containing the set S. this is equivalent to saying that the span of S is the subspace consisting of all linear combinations of the elements of S.

high level objectives of this course

- recognize and formulate convex optimization problems: requires learning about

  * convex sets
  * convex functions
  * equivalent formulations

- identify approximations and relaxations that render a non-convex problem convex

- understand numerical algorithms and software that solve optimization problems

convex optimization [Prof. Stephen Boyd @ Stanford].

- a general optimization problem can be written as

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq b_i \quad i=1,\dots,m \end{aligned}$$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the optimization variable

$f_0 : \mathbb{R}^n \to \mathbb{R}$ is the objective function

$f_i : \mathbb{R}^n \to \mathbb{R}$, $i=1,\dots,m$, define the constraint set and are called constraint functions

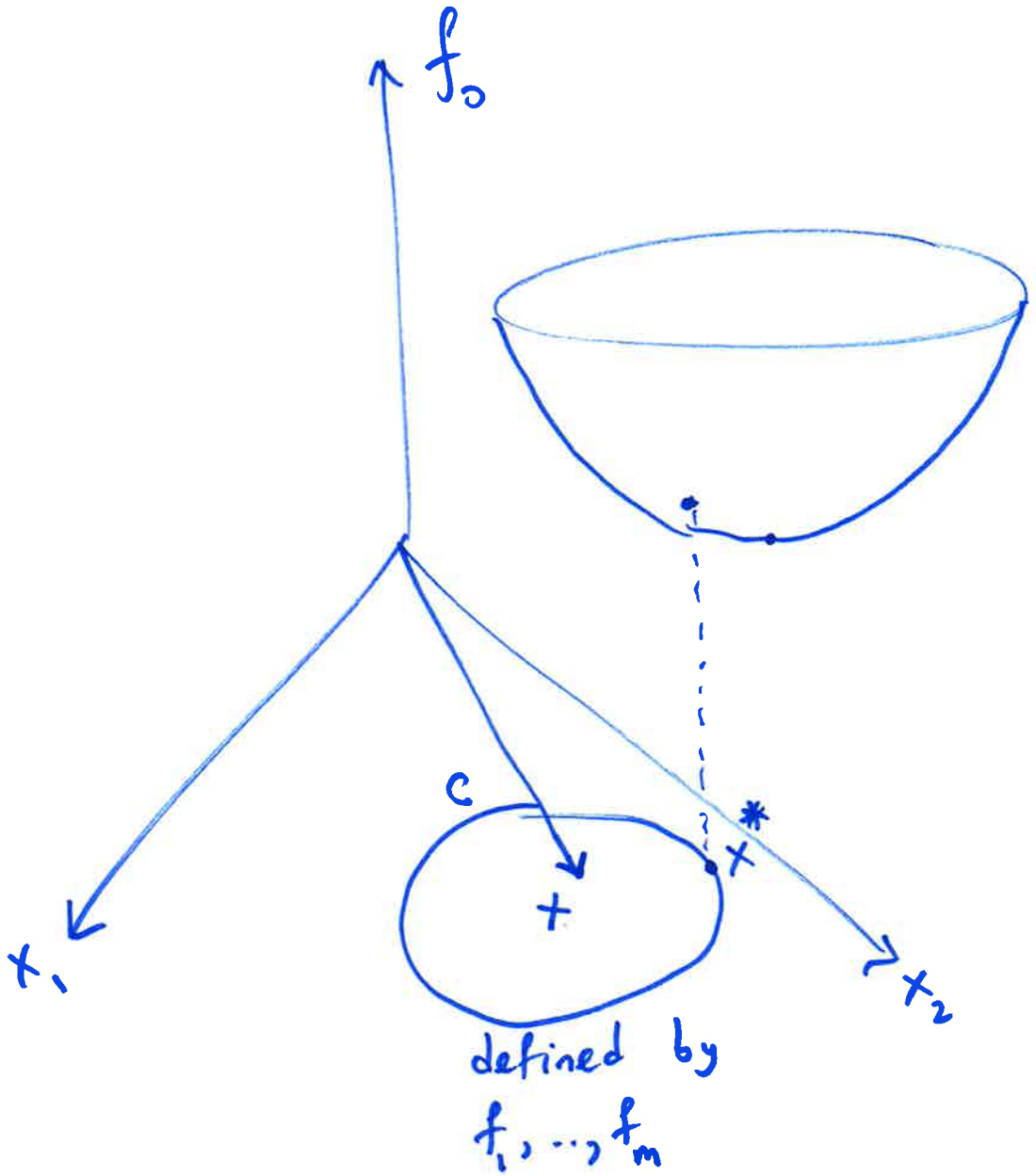
$b_i \in \mathbb{R}$, $i=1,\dots,m$, are the binding parameters for the f_i.

x^* is called optimal (or a solution) if for any z that satisfies $f_1(z) \le b_1, \ldots, f_m(z) \le b_m$ we have $f_0(x^*) \le f_0(z)$.

- in this course, we consider convex optimization problems, where the objective & the constraint functions are convex, i.e., for all $i = 0, 1, \ldots, m$

$$f_i(\theta_1 x_1 + \theta_2 x_2) \le \theta_1 f_i(x_1) + \theta_2 f_i(x_2)$$

for all $x_1, x_2 \in \mathbb{R}^n$, and all $\theta_1, \theta_2 \in \mathbb{R}$, with $\theta_1 + \theta_2 = 1$ & $\theta_1, \theta_2 \ge 0$



minimize $f_0(x)$

subject to $f_i(x) \leq b_i$

$i = 1, \ldots, m$

with functions $f_0, \ldots, f_m$ that are all convex

$$f_i(\theta_1 x_1 + \theta_2 x_2) \leq \theta_1 f_i(x_1) + \theta_2 f_i(x_2)$$

$\forall\, x_1, x_2$ &

$\forall\, \theta_1, \theta_2 \geq 0,\ \theta_1 + \theta_2 = 1$

- some examples (of convex optimization problems)

  - least squares $\min. \ \|Ax-b\|_2^2$

  - least norm $\min. \ \|x\|_2^2$
    $s.t. \ Ax=b$

  - linear program $\min. \ c^T x$
    $s.t. \ a_i^T x \leq b_i \quad i=1,\dots,m$

  - semidefinite program (SDP) $\min. \ \text{trace}(C^T X)$
    $s.t. \ \text{LMI in } X.$

- motivating example: the "dairy" problem

① 22 gallons of milk per week

② 2 gallons of milk → 1 kilogram of butter

③ 3 gallons of milk → 1 gallon of ice cream

④ refrigerator space (to store butter) is unlimited

⑤ freezer space (to store ice cream) is 6 gallons

⑥ 1 hour of work → 4 gallons of ice cream

⑦ 1 hour of work → 1 kilogram of butter

⑧ total hours of work per week is 6

⑨ profit for ice cream is $5 per gallon

⑩ profit for butter is $4 per kilogram

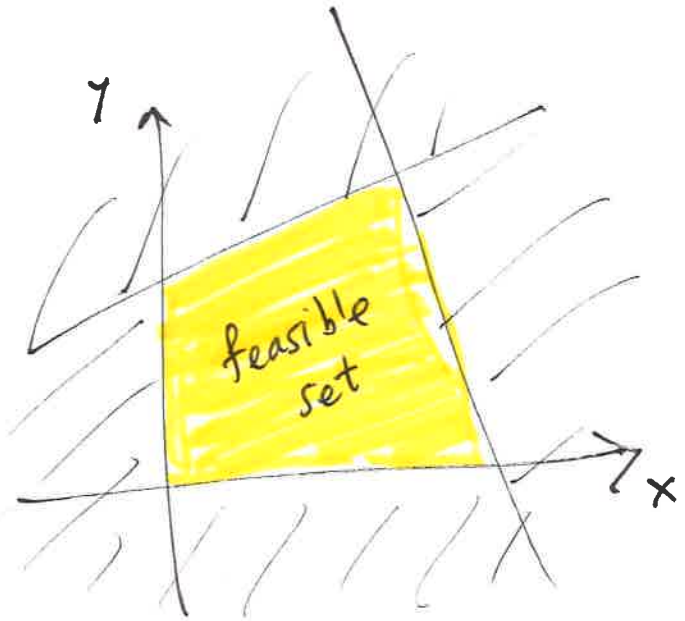
how much butter & ice cream should we make to maximize profit?

---

x: gallons of ice cream

y: kilograms of butter

$J(x,y)$: total profit.



$$\underset{x,y}{\text{maximize}} \quad J(x,y) := 5x + 4y \qquad \leftarrow ⑨ \& ⑩ \quad \text{(profit)}$$

subject to

$$3x + 2y \leq 22 \qquad \leftarrow ①, ②, ③ \quad \text{(resource constraints)}$$

$$x \leq 6 \qquad \leftarrow \cancel{④}, ⑤ \quad \text{(storage constraints)}$$

$$\frac{x}{4} + y \leq 6 \qquad \leftarrow ⑥, ⑦, ⑧ \quad \text{(time constraints)}$$

$$x \geq 0, \; y \geq 0 \qquad \text{(feasibility)}$$

- the diet problem:

x_j: amount of food j

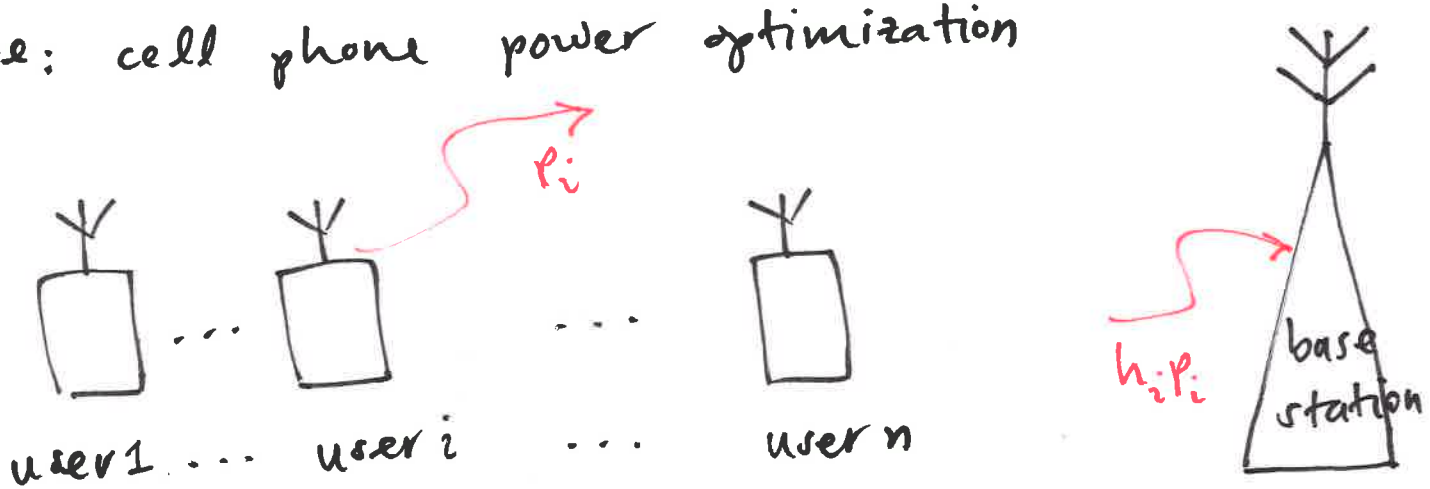
F_{ij}: amount of nutrient i in food j

r_i: minimum daily requirement of nutrient i

c_j: cost of serving unit amount of food j

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Fx \geq r \\ & x \geq 0 \end{aligned}$$

- example: cell phone power optimization



- user i transmits with power p_i.
- attenuation factor h_i (for p_i)
  (actual power received at base station for user i is $h_i p_i$)
- for reliable communication, the signal-to-interference ratio for user i must exceed γ_i.
- interference for user i from all other users is $\sum_{j \neq i} h_j p_j$

it is desired to find p_i, $i=1,\dots,n$, that achieve reliable communication, but minimize total power usage.

$$\begin{aligned} \text{minimize} \quad & p_1 + \cdots + p_n \\ \text{subject to} \quad & p_i \geq 0 \qquad i=1,\dots,n \\ & \frac{h_i p_i}{\sum_{j \neq i} h_j p_j} \geq \gamma_i \qquad i=1,\dots,n \end{aligned}$$

- why convex optimization?
  - guaranteed that an optimal point is globally optimal
  - can be solved efficiently (though almost never analytically or closed-form)

- our aim:

  recognize convex functions & convex constraints in order to identify / formulate convex optimization problems

  (identifying convex problems, or reformulating existing problems as convex ones, can be challenging)

* Convex sets

- lines: suppose $x_1 \neq x_2$ are two points in $\mathbb{R}^n$. The points

$$y = \theta_1 x_1 + \theta_2 x_2 \qquad \theta_1 + \theta_2 = 1, \quad \theta_1, \theta_2 \in \mathbb{R}$$

form the line passing through x_1 and x_2.



equivalently $y = \theta x_1 + (1-\theta) x_2, \quad \theta \in \mathbb{R}$.

(yet another interpretation: $y = x_2 + \theta(x_1 - x_2)$, so that y is sum of base point x_2 and direction $x_1 - x_2$ scaled by θ.)

- affine sets: a set $C \subseteq \mathbb{R}^n$ is affine if the line thru any two points in C lies in C, i.e.,

  for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 + \theta_2 = 1$

  we have $\theta_1 x_1 + \theta_2 x_2 \in C$.

(C contains the linear combination of any two points in C with coefficients that sum to one.)

- we refer to $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1 + \cdots + \theta_k = 1$ as an affine combination of $x_1, \ldots, x_k$. ($\theta_1, \ldots, \theta_k \in \mathbb{R}$)

- an affine set C contains every affine combination of its points

  $x_1, \ldots, x_k \in C$, $\theta_1 + \cdots + \theta_k = 1 \rightarrow \theta_1 x_1 + \cdots + \theta_k x_k \in C$.

review of last lecture

- some ~~convex~~ sets

  - line: given x_1, x_2, $\{y \mid y = \theta_1 x_1 + \theta_2 x_2,\ \theta_1 + \theta_2 = 1,\ \theta_1, \theta_2 \in \mathbb{R}\}$

    (equivalently $y = \theta x_1 + (1-\theta) x_2,\ \theta \in \mathbb{R}$)

  - affine set: C is affine if line through any two points in C lies in C.

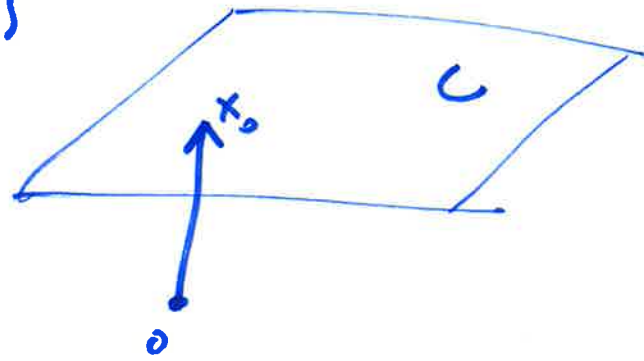
- affine combination: $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1 + \cdots + \theta_k = 1$ is an affine combination of $x_1, \ldots, x_k$

  (compare with linear combination)

- if C is an affine set and $x_0 \in C$ then

$$V = C - x_0 = \{ x - x_0 \mid x \in C \}$$

is a subspace.



conversely, the affine set C can be expressed as

$$C = V + x_0 = \{ v + x_0 \mid v \in V \}$$

i.e., as a "subspace + offset."

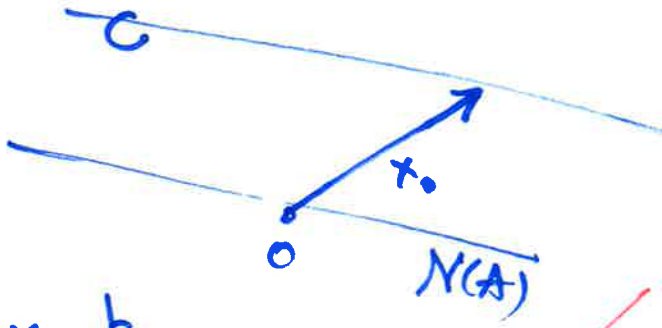
• example: the solution set of linear equations

$$C = \{ x \mid Ax = b \} \qquad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m$$

is an affine set. to see this, suppose $x_1, x_2 \in C$ i.e., $Ax_1 = b$, $Ax_2 = b$. then for any θ

$$\begin{aligned} A(\theta x_1 + (1-\theta) x_2) &= \theta A x_1 + (1-\theta) A x_2 \\ &= \theta b + (1-\theta) b \\ &= b. \end{aligned}$$

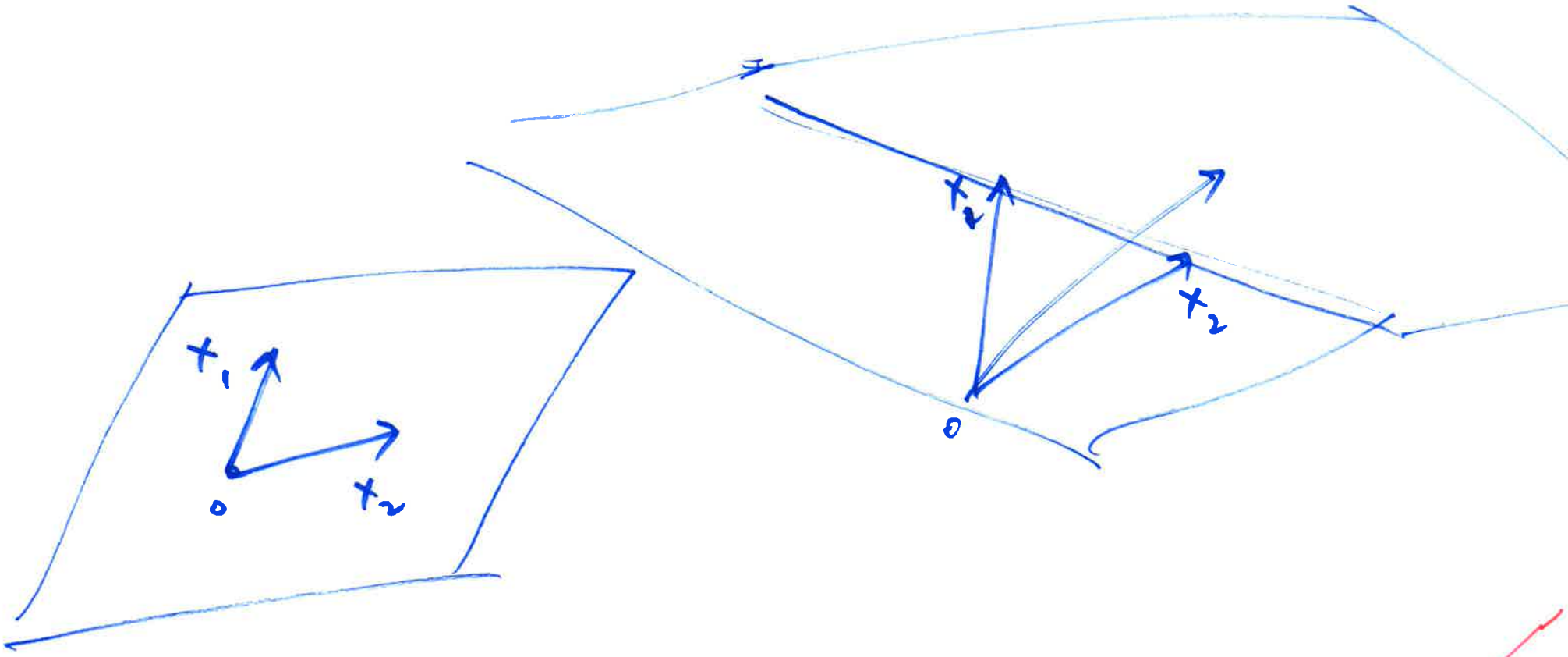
which means that the affine combination $\theta x_1 + (1-\theta) x_2$ is also in C.



i.e., $C = N(A) + x_0$ $\qquad Ax_0 = b$

- affine hull: the set of all affine combinations of points in a set C is called the affine hull of C

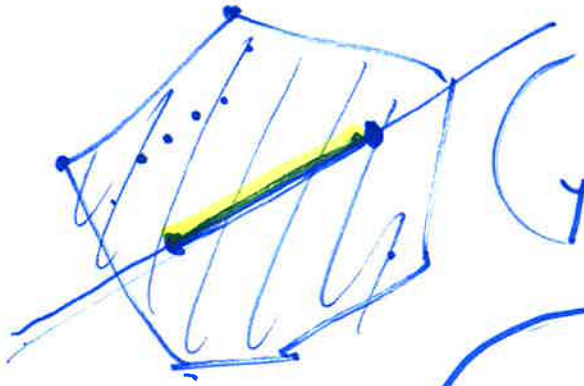
$$\text{aff } C = \left\{ \theta_1 x_1 + \cdots + \theta_k x_k \;\middle|\; x_1, \ldots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1 \right\}$$



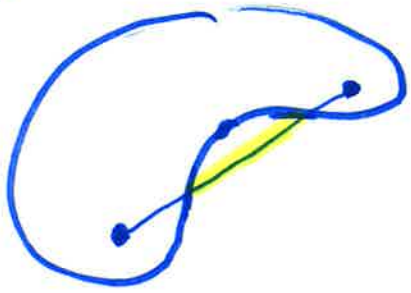
- convex sets: a set C is convex if the line segment between any two points in C lies in C, i.e., for any $x_1, x_2 \in C$ and any $\theta_1, \theta_2 \geq 0$, $\theta_1 + \theta_2 = 1$ we have

$$y = \theta_1 x_1 + \theta_2 x_2 \in C$$

$f_i(x) \leq b_i$



(equivalently $y = \theta x_1 + (1-\theta) x_2 \in C$)



(a set is convex if every point in it can be "seen" by every other point)

4-5

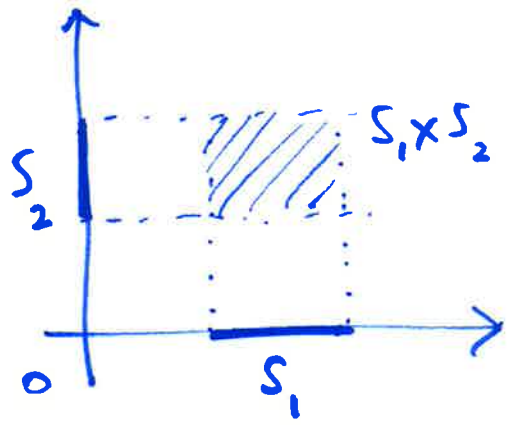
- example: if S_1 & S_2 are convex, then so is their direct (Cartesian) product

$$S_1 \times S_2 = \{ (x_1, x_2) \mid x_1 \in S_1, x_2 \in S_2 \}$$

$\in S_1 \times S_2$ $\qquad$ $\in S_1 \times S_2$

$$\theta (x_1, x_2) + (1-\theta)(y_1, y_2)$$

$$= (\underbrace{\theta x_1 + (1-\theta) y_1}_{\in S_1} , \underbrace{\theta x_2 + ((1-\theta) y_2}_{\in S_2})$$

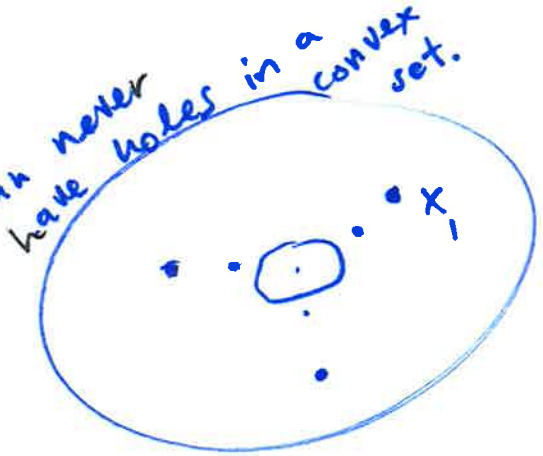


$\downarrow$

$$\theta(x_1, x_2) + (1-\theta)(y_1, y_2) \in S_1 \times S_2$$

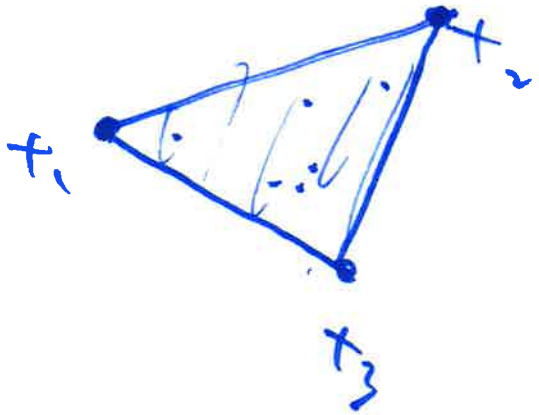
- we call a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$, $\theta_1 + \cdots, \theta_k = 1$ and $\theta_i \geq 0$ for all i, a convex combination of the points $x_1, \ldots, x_k$. a set is convex iff it contains every convex combination of its points.

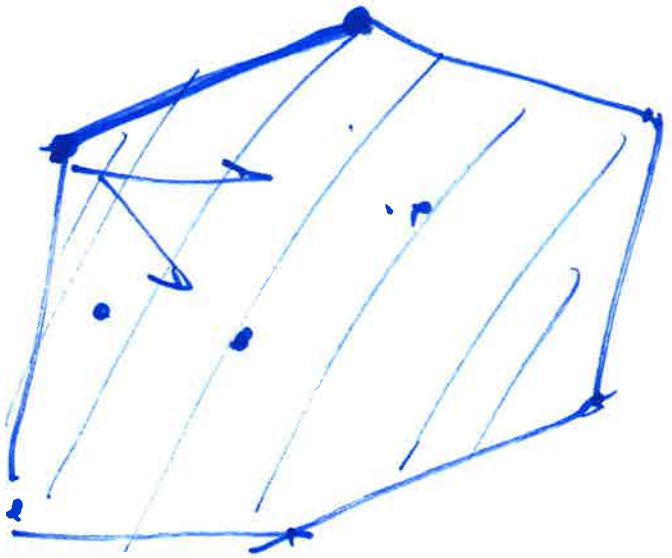
(can think of convex combination as weighted average)

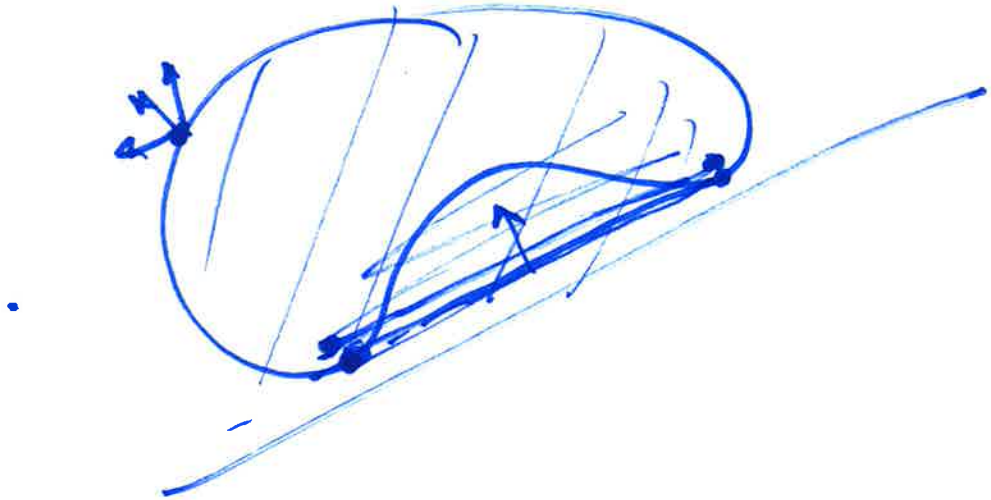


- convex hull: the convex hull of a set C is the set of all convex combinations of points in C

$$\text{conv } C = \left\{ \theta_1 x_1 + \cdots + \theta_k x_k \;\middle|\; x_i \in C \text{ } \forall i,\ \theta_i \geq 0 \text{ } \forall i,\ \sum_i \theta_i = 1 \right\}$$

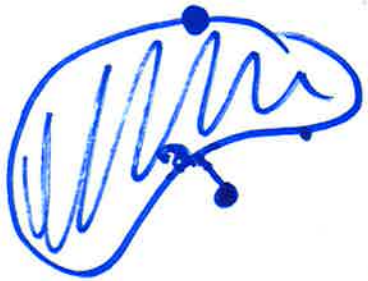

x1
x2
x3





min. $f_0(x)$

s.t.



$x \in \{0,1\}^n$

$f_0(x^*)$

$\geq$

min. $f_0(x)$

s.t.



$x \in [0,1]^n$

relaxed.

$f_0(y^*)$

review of last lecture

- affine hull: $\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \ldots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1\}$

- convex set: if $x_1, x_2 \in C$ then

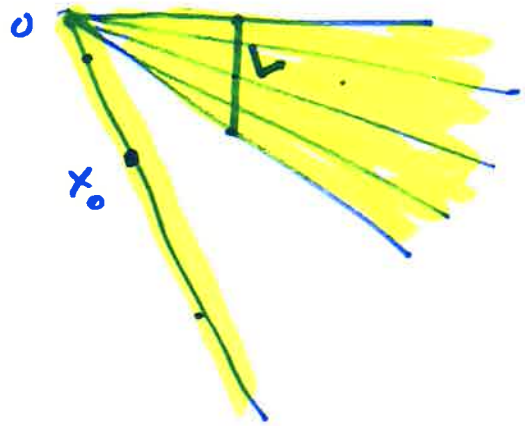
  $\underbrace{\theta_1 x_1 + \theta_2 x_2}_{\theta x_1 + (1-\theta) x_2,\ 1 \geq \theta \geq 0} \in C$ for any $\theta_1 + \theta_2 = 1$, $\theta_1, \theta_2 \geq 0$

  (a set is convex iff its intersection with any line is convex)

- convex combination: $\theta_1 x_1 + \cdots + \theta_k x_k$, $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$ $\forall i$

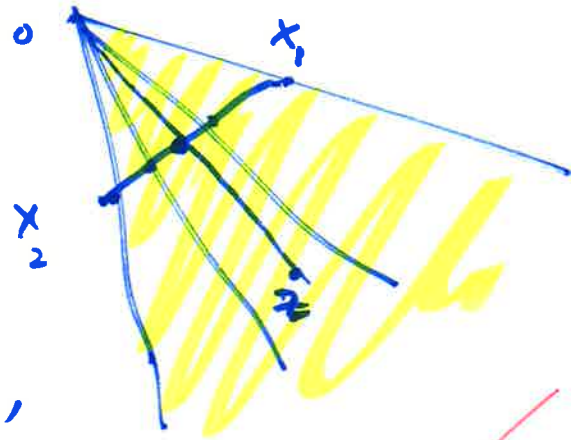
- convex hull (of C): set of all convex combinations of points in C.

- cone: a set C is called a cone if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$.



- convex cone: a set C is a convex cone if it's convex and a cone, which means that for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$ we have

$$\theta_1 x_1 + \theta_2 x_2 \in C$$



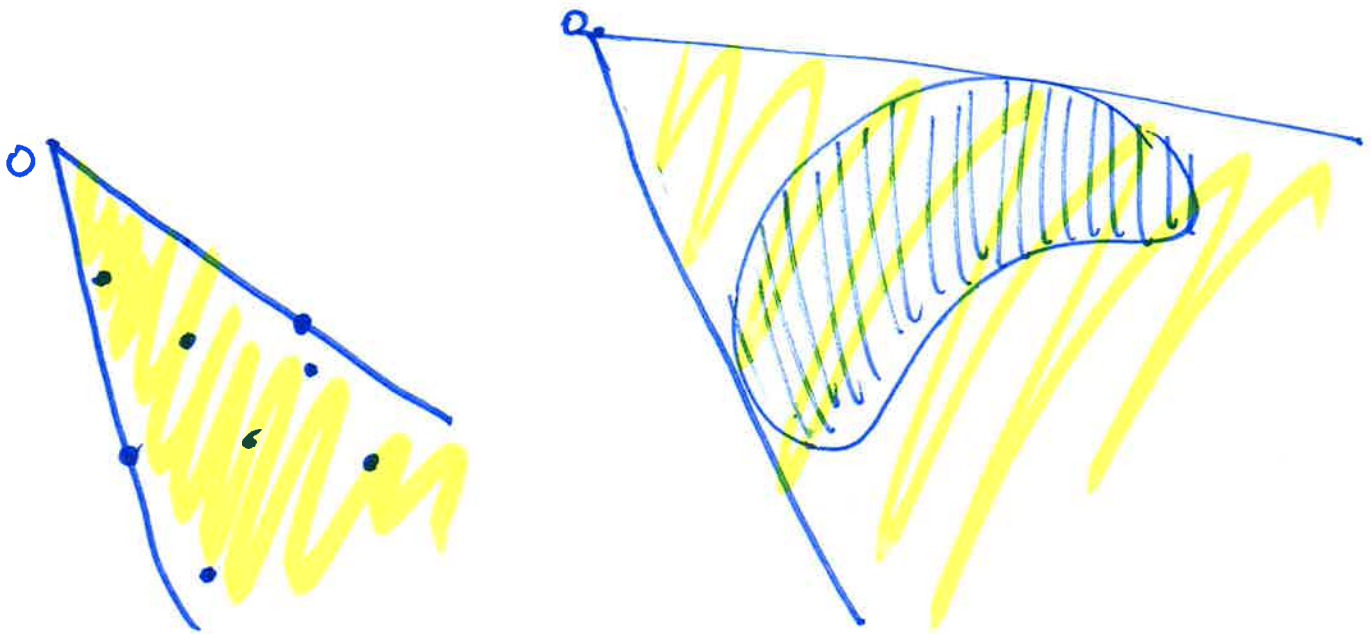
(can think of any point z as $z = \theta(\alpha_1 x_1 + \alpha_2 x_2)$, where $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 \geq 0$, $\theta \geq 0$ so that z is a conv. comb. of x_1, x_2 & a scaling.)

5.2

- a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1, \ldots, \theta_k \geq 0$ is called a conic combination of $x_1, \ldots, x_k$.

- conic hull: the set of all conic combinations of points in C

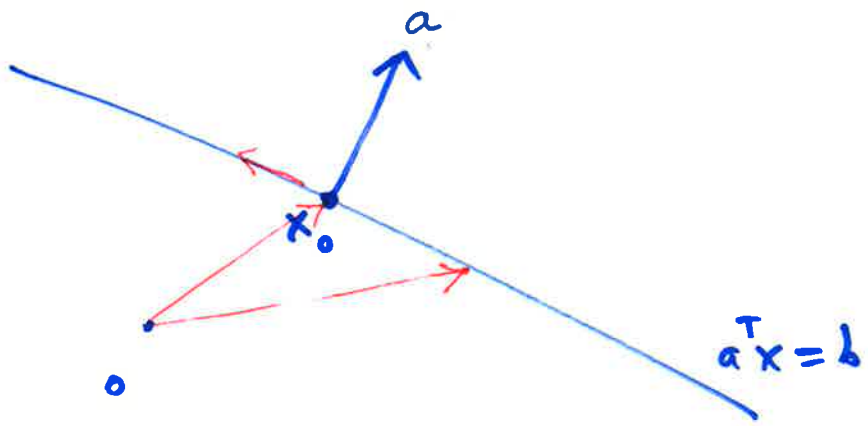
$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C,\ \theta_i \geq 0,\ i = 1, \ldots, k\}.$$



- hyperplane: a hyperplane is a set of the form

$$\{x \mid a^T x = b\} \qquad (a \neq 0)$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.



equivalently, we can represent a hyperplane as

$$\{x \mid a^T(x - x_0) = 0\} \quad \text{or} \quad x_0 + a^{\perp}$$

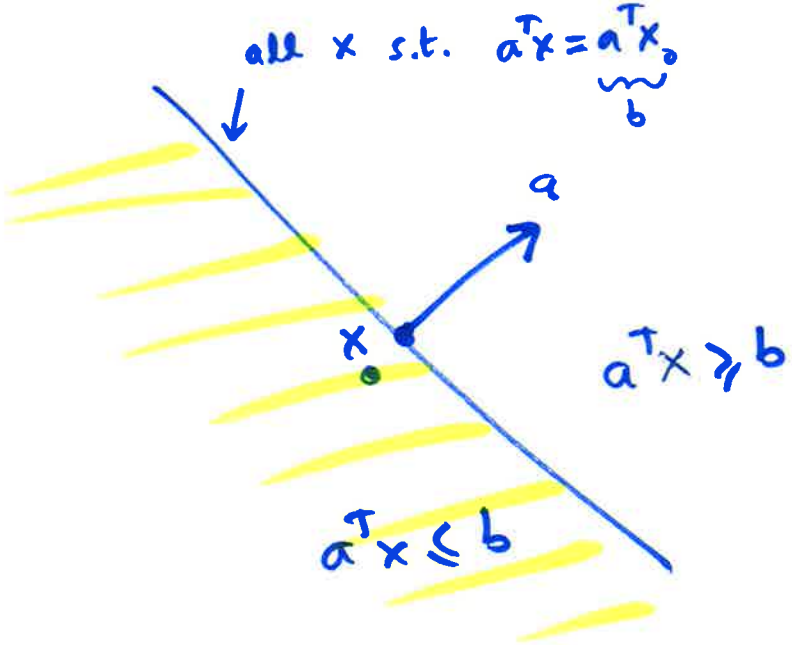
where x_0 is such that $a^T x_0 = b$, & $a^{\perp} = \{v \mid a^T v = 0\}$.

(the hyperplane consists of an offset x_0 plus all vectors orthogonal to a)

- halfspace: a (closed) halfspace is a set of the form

$$\{x \mid a^T x \leq b\} \quad \text{or} \quad \{x \mid a^T(x - x_0) \leq 0\}$$

where $a^T x_0 = b$.



- a hyperplane divides $\mathbb{R}^n$ into two half spaces.

- open half space: $\{x \mid a^T x < b\}$.

- Euclidean ball: a ball in $\mathbb{R}^n$ has the form

$$B(x_c, r) = \{ x \mid \|x - x_c\|_2 \le r \}$$

$$= \{ x \mid (x - x_c)^T (x - x_c) \le r^2 \}$$

where $r > 0$.

x_c: center

r: radius

equivalently,

$$B(x_c, r) = \{ x_c + ru \mid \|u\|_2 \le 1 \}$$

a Euclidean ball is convex.

norm $\|\cdot\|$:

- $\|x\| \ge 0$; $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\alpha x\| = |\alpha| \, \|x\|$ $\alpha \in \mathbb{R}$
- $\|x + y\| \le \|x\| + \|y\|$

- norm ball: for any norm the ball with center x_c and radius r is $\{ x \mid \|x - x_c\| \le r \}$

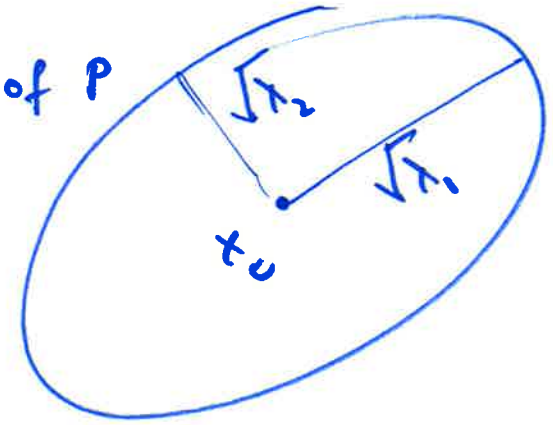
- **ellipsoid:** an ellipsoid in $\mathbb{R}^n$ has the form

$$E = \{ x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1 \}$$

where $P = P^T > 0$ (P is symmetric & pos. def.)

x_c: center

λ_i: i^{th} eig of P



equivalently,

$$E = \{ x_c + Au \mid \|u\|_2 \leq 1 \}$$

$$A = P^{1/2} > 0$$

---

aside:

$$A > 0 \iff x^T A x > 0 \quad \forall x \neq 0 \iff \lambda_i > 0 \quad \forall i$$

$$A \geq 0 \iff x^T A x \geq 0 \quad \forall x \iff \lambda_i \geq 0 \quad \forall i$$

$$A = A^T$$

proof using diagonalization.

aside:

$$A = U \Lambda U^T$$

$$x^T A x = \underbrace{x^T U}_{y^T} \Lambda \underbrace{U^T x}_{y}$$

$$= y^T \Lambda y = \sum y_i^2 \lambda_i \geq 0 \quad \forall y$$

iff

$\lambda_i \geq 0 \ \forall i$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

aside:

$$P = U \Lambda U^T$$

$$P^q := U \Lambda^q U^T$$

$$\Lambda^q = \begin{bmatrix} \lambda_1^q & & \\ & \ddots & \\ & & \lambda_n^q \end{bmatrix}$$

$$P^{1/2} := U \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix} U^T.$$

- norm cone: norm cone associated with the norm $\|\cdot\|$ is

$$C = \{ (x,t) \mid \|x\| \le t \} \subseteq \mathbb{R}^{n+1}$$

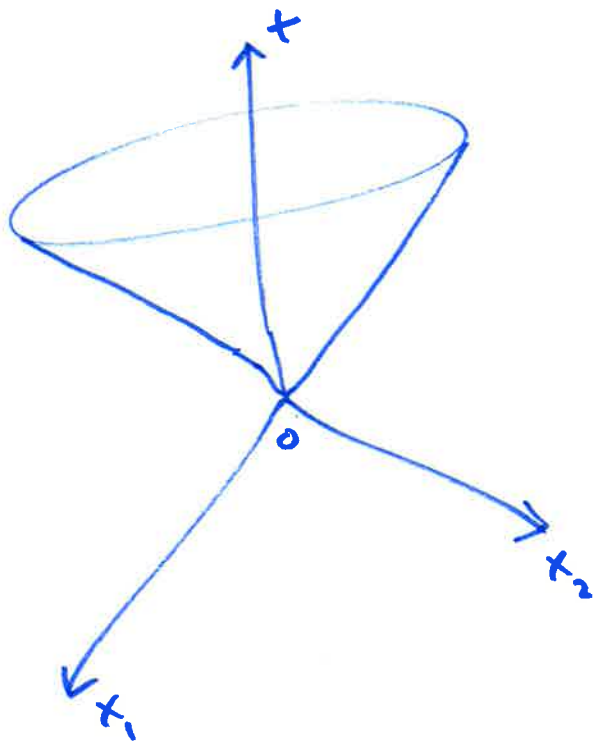
$x \in \mathbb{R}^n$

$t \in \mathbb{R}$.

$t \ge 0$

the second-order cone is

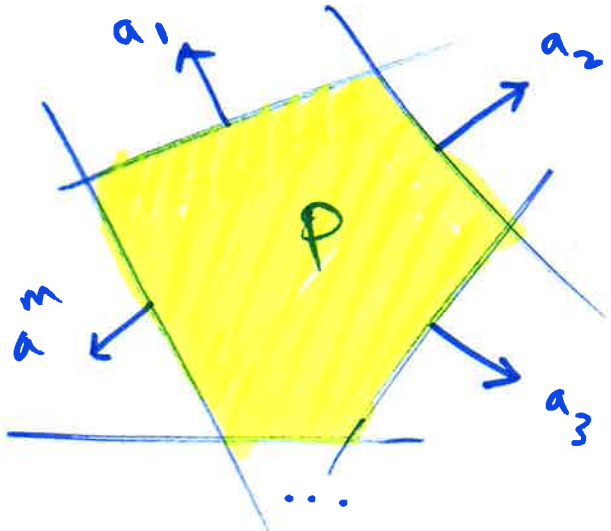
$$C = \{ (x,t) \mid \|x\|_2 \le t, \ t \ge 0 \}.$$



- polyhedron : a polyhedron is defined as the solution set of a finite number of linear equalities & inequalities

$$P = \{ x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m, \quad c_i^T x = d_i, \quad i = 1, \dots, p \}$$

a polyhedron is the intersection of a finite number of half spaces & hyperplanes.



aside : the finiteness is important; the intersection of an ∞ number of half spaces may not be polyhedron.

we will hereafter use the compact notation

$$P = \{ x \mid Ax \preceq b,\ Cx = d \}$$

where

$$A = \begin{bmatrix} -a_1^T- \\ \vdots \\ -a_m^T- \end{bmatrix}, \quad C = \begin{bmatrix} -c_1^T- \\ \vdots \\ -c_p^T- \end{bmatrix}$$

and the symbol $\preceq$ denotes componentwise/entrywise inequality:

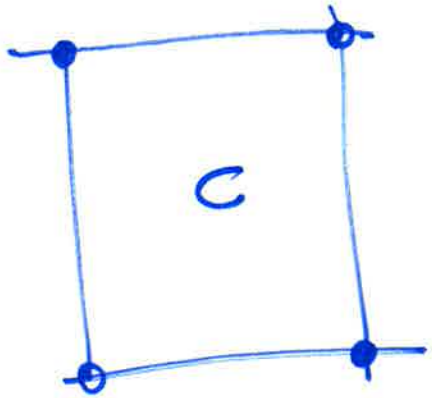
$$u \preceq v \iff u_i \leq v_i \quad \forall i$$

($\prec$ $\quad u \prec v \iff u_i < v_i \quad \forall i$)

aside:

$$C = \{x \mid \|x\|_\infty \le 1\}$$

$$= \{x \mid |x_i| \le 1 \quad \forall i\}$$



$x_i \le 1$ & $x_i \ge -1$

$-x_i \le 1$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ entry}$$

$e_i^T x = x_i$

$x_i \le 1 \rightarrow e_i^T x \le 1$

$-x_i \le 1 \rightarrow -e_i^T x \le 1$

$$C = \{x \mid \pm e_i^T x \le 1 \quad \forall i\}.$$ (2n)

description requires $2n$ linear inequalities

$$C = \text{conv}\{\text{corners of } 1\text{-ball in } \alpha\text{-norm}\}.$$

$$= \text{conv}\{\text{corners of cube in } \mathbb{R}^n\}.$$

$$= \text{conv}\{v_1, \ldots, v_{2^n}\}.$$

2^n

description requires 2^n points

$n^2 m$

complexity of numerically solving convex optimization problem, where n is number of optz var & m is # of constr.

- positive semidefinite cone: the set of symmetric positive semidefinite matrices

$$S_+^n = \{ X \mid X \succeq 0 \}$$

is a convex cone

$A, B \in S_+^n$ then $\theta_1 A + \theta_2 B \in S_+^n$ for all $\theta_1, \theta_2 \geq 0$

$$x^T(\theta_1 A + \theta_2 B)x = \theta_1 \underbrace{x^T A x}_{\geq 0} + \theta_2 \underbrace{x^T B x}_{\geq 0} \geq 0$$

- example: pos. semidef. cone in S^2:

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_+^2 \iff x \geq 0,\ z \geq 0,\ xz \geq y^2$$

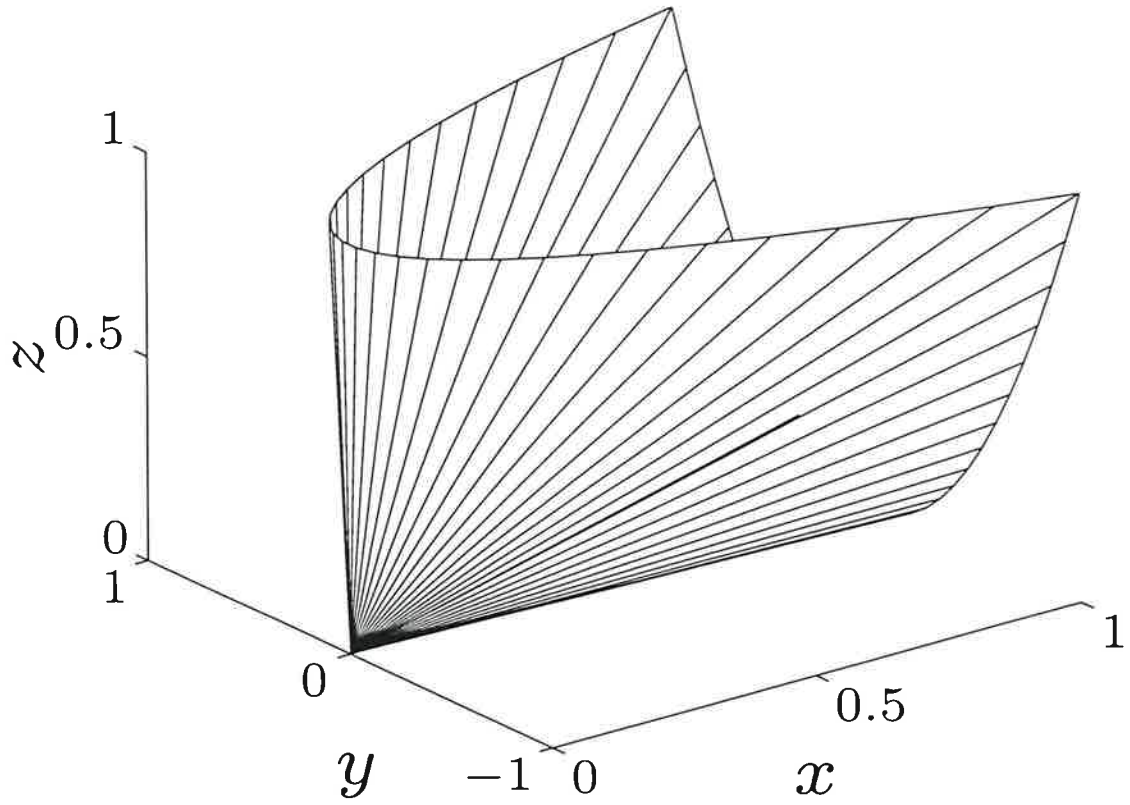
aside: notation $S^n := \{ X \in \mathbb{R}^{n \times n} \mid X = X^T \}$; $S_+^n := \{ X \in S^n \mid X \succeq 0 \}$; $S_{++}^n := \{ X \in S^n \mid X \succ 0 \}$

$x \geq 0 \qquad z \geq 0$

$xz - y^2 \geq 0$

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$$

$y^2 \leq xz.$



operations that preserve convexity (of sets)

- methods for establishing convexity of set C

  ① apply a definition

  e.g., $x_1, x_2 \in C$, $0 \le \theta \le 1 \rightarrow \theta x_1 + (1-\theta) x_2 \in C$

  ② show that C is obtained from simple convex sets (e.g., hyperplanes, half spaces, norm balls, etc) by operations that preserve convexity

  - intersection
  - affine functions
  - ⋮

- intersection: if S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex.

- example: a polyhedron is the intersection of halfspaces and hyperplanes, which are convex, and is therefore a convex set.

- example: the positive semidefinite cone $\mathbb{S}_+^n$ is a convex set.

  first consider $\left\{ X \in \mathbb{S}^n \,\middle|\, v^T X v \geq 0 \right\}$ with $v \neq 0$, which is a halfspace. (for each $v \neq 0$, $v^T X v$ is a linear function of X so that the above set is a halfspace)

---

aside: $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$ $\quad v^T X v = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{pmatrix} x & y \\ y & z \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1^2 x + 2 v_1 v_2 y + v_2^2 z$

$$= \begin{bmatrix} v_1^2 & 2v_1 v_2 & v_2^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq 0$$

$$\bigcap_{v \neq 0} \left\{ X \in \mathbf{S}^n \;\middle|\; v^T X v \geq 0 \right\}$$

is the intersection of an infinite number of halfspaces and therefore is convex.

- affine function mapping: suppose $f: \mathbf{R}^n \to \mathbf{R}^m$ is an affine function, i.e., it is the sum of a linear function and a constant, $f(x) = Ax + b$, $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$.

suppose $S \subseteq \mathbf{R}^n$ is convex and $f: \mathbf{R}^n \to \mathbf{R}^m$ is affine.

---

aside:

$$\mathbf{S}^n = \left\{ X \in \mathbf{R}^{n \times n} \;\middle|\; X = X^T \right\}$$
$$\mathbf{S}^n_+ = \left\{ X \in \mathbf{S}^n \;\middle|\; X \geq 0 \right\}$$
$$\mathbf{S}^n_{++} = \left\{ X \in \mathbf{S}^n \;\middle|\; X > 0 \right\}$$

$$\mathbf{R}^n$$
$$\mathbf{R}^n_+ = \left\{ x \in \mathbf{R}^n \;\middle|\; x \geq 0 \right\}$$
$$\mathbf{R}^n_{++} = \left\{ x \in \mathbf{R}^n \;\middle|\; x > 0 \right\}$$

then the image of S under f is convex

$$f(S) = \{ f(x) \mid x \in S \}$$
$$= \{ Ax+b \mid x \in S \}.$$

similarly, the inverse image of S under f is convex

$$f^{-1}(S) = \{ x \mid f(x) \in S \}$$
$$= \{ x \mid Ax+b \in S \},$$

where $f: \mathbb{R}^k \to \mathbb{R}^n$

- examples: if $S \subseteq \mathbb{R}^n$ is convex, $\alpha \in \mathbb{R}$, $b \in \mathbb{R}^n$, then the sets αS and $S + b$ are convex

$$\alpha S = \{ \alpha x \mid x \in S \} \qquad A = \alpha I,\ b = 0$$

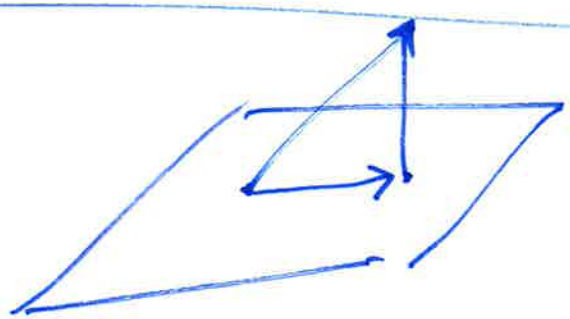
$$S + b = \{ x + b \mid x \in S \} \qquad A = I$$

if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then the projection of S onto some of its coordinates is convex, i.e.,

$$T = \{ x_1 \in \mathbb{R}^m \mid \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S \text{ for some } x_2 \in \mathbb{R}^n \}$$

is convex.

$$A = [I \ \ 0], \quad b = 0$$



$$A = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

if S_1 and S_2 are convex, then $S_1 + S_2$ is convex, i.e.,

$$S_1 + S_2 = \{ x + y \mid x \in S_1 \text{ and } y \in S_2 \}$$

is convex. to see this, recall that if S_1 and S_2 are convex, then

$$S_1 \times S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 \in S_1, \; x_2 \in S_2 \right\}$$

is a convex set. the image of $S_1 \times S_2$ under the linear function

$$f(x_1, x_2) = x_1 + x_2$$

gives the sum $S_1 + S_2$. this corresponds to $A = [I \;\; I]$, $b = 0$.

- the ellipsoid $E = \{ x \mid (x-x_c)^T P^{-1} (x-x_c) \leq 1 \}$ with $P > 0$ is convex. Let the unit ball be defined as

$$B_1 = \{ u \mid \|u\|_2 \leq 1 \}.$$

$$f(u) = P^{1/2} u + x_c$$

$$f(B_1) = \{ f(u) \mid u \in B_1 \}$$

$$= \{ P^{1/2} u + x_c \mid \|u\|_2 \leq 1 \}$$

image of unit ball under f.

$$g(x) = P^{-1/2} (x - x_c)$$

$$g^{-1}(B_1) = \{ x \mid g(x) \in B_1 \}$$

$$= \{ x \mid \| P^{-1/2} (x - x_c) \|_2 \leq 1 \}$$

inverse image of unit ball under g.

- recall

example: the convex set described by the solution of a linear matrix inequality (LMI) $\{x \mid A(x) \preceq B\}$, with $A(x) = x_1 A_1 + \cdots + x_n A_n$, $A_i, B \in S^m$, can be expressed as the inverse image of the positive semidefinite cone S^m_+ under the affine function $f(x) = B - A(x)$ $(f: \mathbb{R}^n \to S^m)$

$$f^{-1}(S^m_+) = \{x \mid \underbrace{f(x) \in S^m_+}_{B - A(x) \succeq 0}\}$$

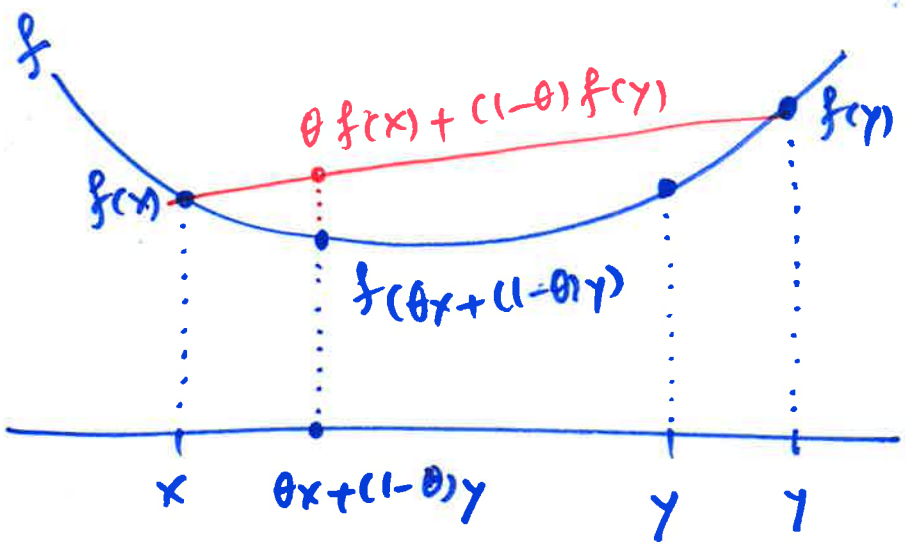
aside: $A(x) = x_1 A_1 + \cdots + x_n A_n \preceq B$ $(B, A_i \in S^m)$ is called an LMI in x.

convex functions

- convex function: a function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom} f$ is convex and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathrm{dom} f$ and $0 \leq \theta \leq 1$.



- f is concave if $-f$ is convex (and $\mathrm{dom} f$ is a convex set.)

implied by convexity of $-f$.

- f is strictly convex if $\mathbf{dom} f$ is convex and

$$f(\theta x + (1-\theta) y) < \theta f(x) + (1-\theta) f(y) \qquad x \neq y$$

$x, y \in \mathbf{dom} f$ and $0 < \theta < 1$.

- examples:
  - affine function: $a^T x + b$ is convex on $\mathbb{R}^n$ for any $a \in \mathbb{R}^n$, $b \in \mathbb{R}$
  - exponential: e^{ax} is convex on $\mathbb{R}$ for any $a \in \mathbb{R}$.
  - powers of absolute value: $|x|^p$ is convex on $\mathbb{R}$ for $p \geq 1$
  - negative log: $-\log x$ is convex on $\mathbb{R}_{++}$
  - negative entropy: $x \log x$ is convex on $\mathbb{R}_{++}$

- norms: $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$ for $p \geq 1$

  $\|x\|_\infty = \max_i |x_i|$

  } are all convex on $\mathbb{R}^n$.

- max function: $f(x) = \max_i x_i$ is convex on $\mathbb{R}^n$
- log-sum-exp: $f(x) = \log(e^{x_1} + \cdots + e^{x_n})$ is convex on $\mathbb{R}^n$.
- negative geometric mean: $f(x) = -(x_1 \cdots x_n)^{1/n}$ is convex on $\mathbb{R}^n_{++}$.
- quadratic-over-linear: $f(x,y) = \frac{x^2}{y}$ is convex on $\operatorname{dom} f = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

  $= \mathbb{R} \times \mathbb{R}_{++}$

- affine function of matrix: $f(X) = \text{trace}(A^T X) + b$

  is convex on $\mathbb{R}^{n \times m}$ for all A in $\mathbb{R}^{n \times m}$ and $b \in \mathbb{R}$.

- spectral norm: $f(X) = \|X\|_2 = \sqrt{\lambda_{max}(X^T X)} = \sigma_{max}(X)$

  is convex on $\mathbb{R}^{n \times m}$.

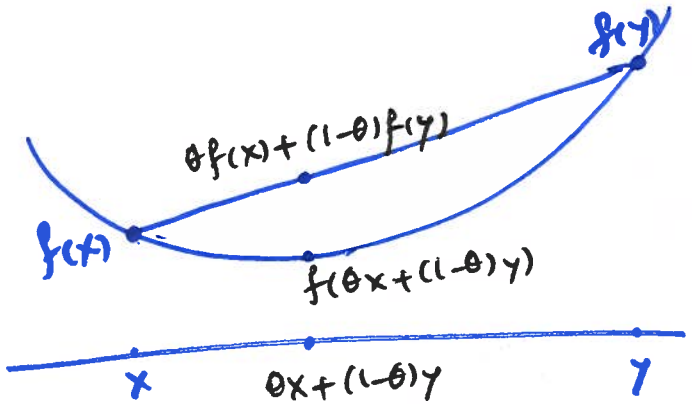
- max eig: $f(X) = \max_i \lambda_i(X)$ is convex on $\mathbb{R}^{n \times n}$.

- negative log-det: $f(X) = -\log \det(X)$ is convex on $\text{dom } f = S^n_{++}$

review of last lecture

- operations that preserve convexity of sets
  - intersection
  - affine function image & inverse image

- convex function $f: \mathbb{R}^n \to \mathbb{R}$



  dom f convex, and

  $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$ $\quad x, y \in \text{dom} f, \quad 0 \le \theta \le 1$

- f strictly convex if

  dom f convex, and

  $f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$ $\quad x, y \in \text{dom} f, \quad 0 < \theta < 1$

- examples

- a function is convex if and only if it is convex when restricted to any line that intersects $\mathbf{dom}\, f$, i.e.,

  for all $x \in \mathbf{dom} f$ and all vectors v,
  the function g defined as

  $$g(t) = f(x + tv)$$

  is convex (in t) on its domain $\mathbf{dom}\, g = \{t \mid x + tv \in \mathbf{dom} f\}$

- review of gradient

x: scalar, f: scalar-valued

$$f(x+\delta x) = f(x) + \underbrace{f'(x)\,\delta x}_{\delta f} + \underbrace{\frac{1}{2} f''(x)\,\delta x^2}_{\delta^2 f} + \text{h.o.t.}$$

δf is first-order variation of f at x

$\delta^2 f$ is second-order variation of f at x

---

x: vector, f: scalar-valued

$$f(x+\delta x) = f(x) + \underbrace{(\nabla f(x))^T \delta x}_{\delta f} + \underbrace{\frac{1}{2}(\delta x)^T \left(\nabla^2 f(x)\right) \delta x}_{\delta^2 f} + \text{h.o.t.}$$

$\nabla f(x)$: "gradient" of f at x

$\nabla^2 f(x)$: "Hessian" of f at x.

$$\delta f = \frac{\partial f}{\partial x_1}\delta x_1 + \cdots + \frac{\partial f}{\partial x_n}\delta x_n$$

$$= (\nabla f)^T \delta x$$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix}_{n\times 1} \qquad \delta x = \begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{bmatrix}_{n\times 1}$$

$$\delta^2 f = \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\delta x_i \delta x_j$$

$$= \frac{1}{2}(\delta x)^T (\nabla^2 f)\delta x$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \\ \vdots & & \frac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix}_{n\times n}$$

$$\nabla^2 f = (\nabla^2 f)^T$$

7-4

X: matrix, f: scalar-valued

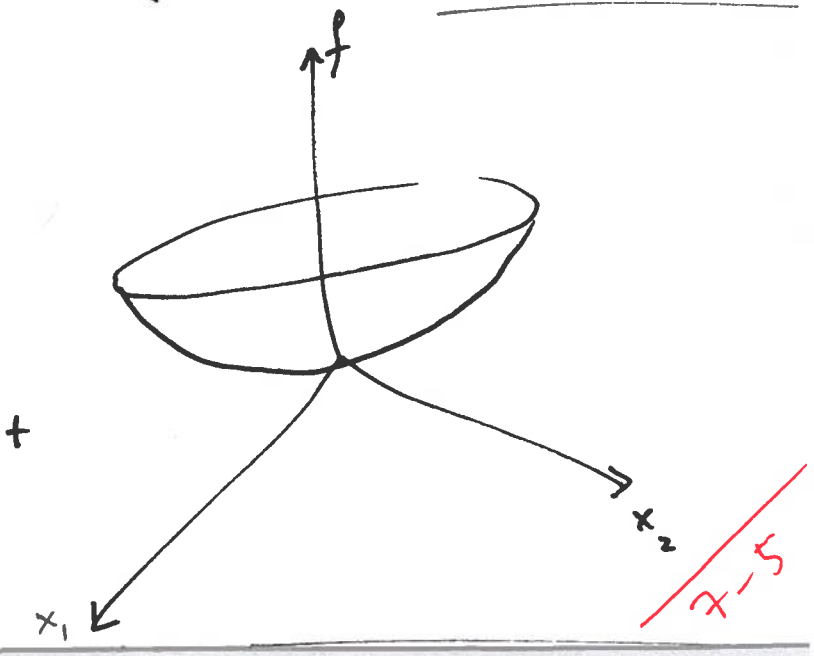
$$f(X+\delta X) = f(X) + \underbrace{\mathrm{tr}\left((\nabla f)^T \delta X\right)}_{\delta f} + \underbrace{\frac{1}{2}\mathrm{tr}\left(H(\delta X, X)\,\delta X\right)}_{\delta^2 f} + \text{h.o.t.}$$

∇f : a matrix $H(\delta X, X)$: a matrix & linear in δX.

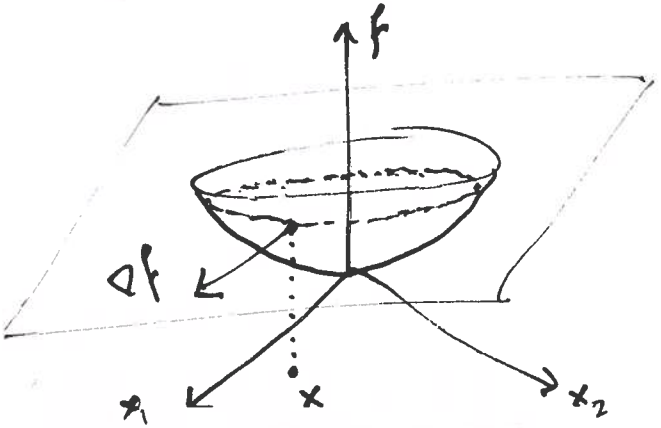
- example: $f(x_1, x_2) = x_1^2 + x_2^2$

$$\nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \qquad \nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

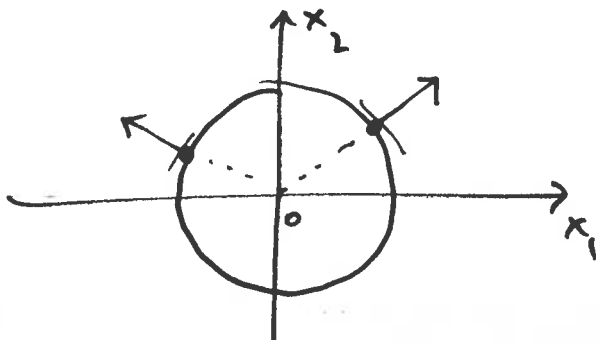
$$\nabla f = 0 \longrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$ unique minimum point of f.



- interpretation of ∇f: the gradient always points in the direction of maximum ascent.



$$\nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$



aside (on gradients):

for any function $f(x)$:

- suppose x_0 is such that $f(x_0) = c$. (x_0 is on c-level curve) then $\nabla f(x_0)$ is orthogonal to the c-level curve at $x = x_0$

- at any point $x = x_0$, the gradient $\nabla f(x_0)$ points in the direction of maximal increase in the value of f. (proof: $\delta f = (\nabla f)^T \delta x$!)

- example: Least-squares objective

$$J(x) = \|Ax-b\|_2^2 = (Ax-b)^T(Ax-b)$$

where we assume that the columns of A are lin. ind.

$$\begin{aligned} J(x+\delta x) &= (Ax + A\delta x - b)^T(Ax + A\delta x - b) \\ &= (Ax-b)^T(Ax-b) + (A\delta x)^T(Ax-b) \\ &\quad + (Ax-b)^T(A\delta x) + (A\delta x)^T(A\delta x) \\ &= \underbrace{(Ax-b)^T(Ax-b)}_{J(x)} + \underbrace{2(Ax-b)^T A\delta x}_{\delta J(x)} + \underbrace{(\delta x)^T A^T A\delta x}_{\delta^2 J(x)} \end{aligned}$$

aside:

$$y^T z = z^T y$$

$$\delta J = (\nabla J)^T \delta x = \left(2A^T(Ax-b)\right)^T \delta x \longrightarrow \nabla J = 2A^T(Ax-b).$$

$$\delta^2 J = \frac{1}{2}(\delta x)^T \nabla^2 J\, \delta x = \frac{1}{2}(\delta x)^T 2A^T A\, \delta x \longrightarrow \nabla^2 J = 2A^T A.$$

$\nabla J(x^*) = 0 \rightarrow A^TAx^* - A^Tb = 0 \rightarrow A^TAx^* = A^Tb \rightarrow \boxed{x^* = (A^TA)^{-1}A^Tb}$

assuming A^TA invertible

$\nabla^2 J(x^*) \overset{?}{\succ} 0 \leftrightarrow$ is A^TA a pos. def. mat. ?

$\leftrightarrow z^T(A^TA)z > 0 \quad \forall z \neq 0$?

$z^TA^TAz = (Az)^T(Az) = \|Az\|_2^2 \geq 0 \rightarrow \begin{cases} \|Az\|_2^2 = 0 \\ \|Az\|_2^2 > 0 \end{cases}$

$\|Az\|_2^2 = 0 \leftrightarrow Az = 0$

↓

this contradicts the lin. ind. of col. of A

$z^TA^TAz > 0 \quad \forall z \rightarrow A^TA \succ 0$

& in particular, A^TA is invertible.

- procedure for finding ∇f & $\nabla^2 f$ of a scalar-valued function, $f(x)$, is as follows:

  - set $x \longrightarrow x + \delta x$
  - expand $f(x+\delta x)$, in δx, around x.
    (i.e., write Taylor expansion of f at x in terms of δx)
  - identify the δf & $\delta^2 f$ terms, and extract ∇f & $\nabla^2 f$.

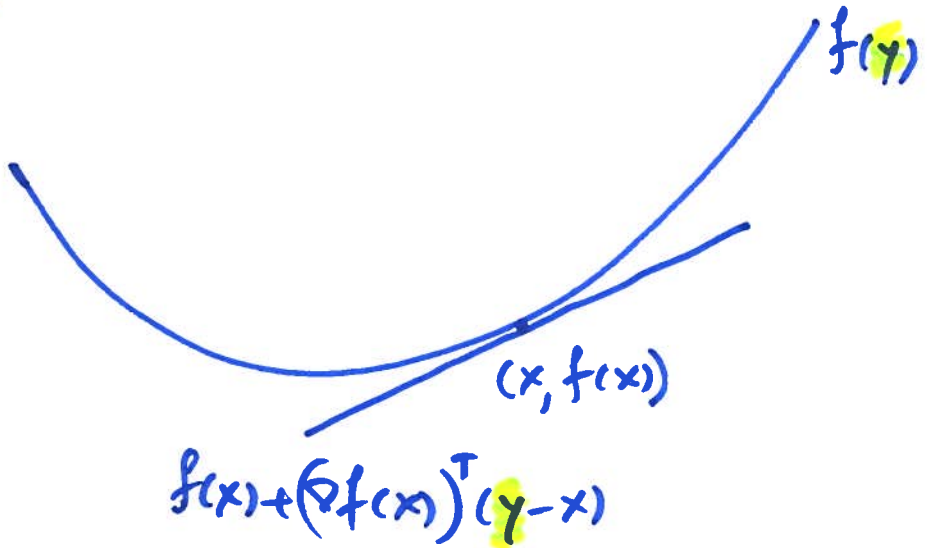
(when the expression for f is simple or given explicitely in terms of x_i, then direct computation of $\frac{\partial f}{\partial x_i}$ & $\frac{\partial^2 f}{\partial x_i \partial x_j}$ may be easier way of getting ∇f & $\nabla^2 f$, as in example on previous slide.)

- first-order conditions for convexity (of a function):

  suppose f is differentiable. then f is convex if and only if $\mathbf{dom}\, f$ is convex and

$$f(y) \geq f(x) + (\nabla f(x))^T (y-x)$$

  holds for all $x, y \in \mathbf{dom}\, f$.



- if $\nabla f(x) = 0$ then for $y \in \mathbf{dom}\, f$ we have $f(y) \geq f(x)$, i.e., x is a global minimizer of f.

- f is strictly convex if and only if $\mathbf{dom}\, f$ is convex and

$$f(y) > f(x) + (\nabla f(x))^T (y-x)$$

for all $x, y \in \mathbf{dom}\, f$, $x \neq y$.

- proof of first-order convexity condition:

  * first consider the case of $n=1$.

($\Rightarrow$) assume that f is convex and $x, y \in \mathbf{dom}\, f$. then $x + \theta(y-x) \in \mathbf{dom}\, f$ for all $0 \leq \theta < 1$, and

$$f(x+\theta(y-x)) \leq (1-\theta) f(x) + \theta f(y)$$

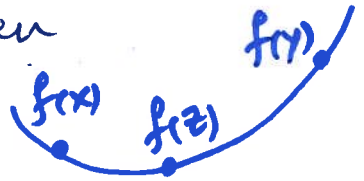
$$\theta f(y) \geq \theta f(x) - f(x) + f(x+\theta(y-x))$$

$$f(y) \geq f(x) + \frac{f(x+\theta(y-x)) - f(x)}{\theta(y-x)} (y-x)$$

taking $\theta \to 0$ gives $f(y) \geq f(x) + f'(x)(y-x)$.

($\Leftarrow$) assume that ! holds, for all

$x, y \in \mathbf{dom}\, f$. for $x \neq y$ let $z = \theta x + (1-\theta) y$, $0 \leq \theta \leq 1$. then



$f(x) \geq f(z) + f'(z)(x-z)$ $\quad$ $f(y) \geq f(z) + f'(z)(y-z)$

therefore (multiplying first inequality by θ, second by $1-\theta$, then adding)

$$\theta f(x) + (1-\theta) f(y) \geq f(z) + f'(z)\big(\underbrace{\theta x - \theta z + (1-\theta) y}_{z} - (1-\theta) z\big)$$

$$= f(\theta x + (1-\theta) y)$$

which implies convexity of f.

* now consider general n

Let $x, y \in \mathbb{R}^n$ and let f be restricted to the line that passes thru them

$$g(t) = f(ty + (1-t)x), \quad g'(t) = \left(\nabla f(ty + (1-t)x)\right)^T (y - x)$$

($\Rightarrow$) first assume f is convex, which implies that g is convex

$$g(t) \geq g(s) + g'(s)(t-s)$$

now set $t = 1$, $s = 0$

$$f(y) \geq f(x) + \left(\nabla f(x)\right)^T (y - x). \quad \uparrow !$$

($\Leftarrow$) assume that $\uparrow$ holds for any x, y so that if

$ty + (1-t)x \in \text{dom } f$

$sy + (1-s)x \in \text{dom } f$

then $\downarrow$

$$f(ty+(1-t)x) \geq f(sy+(1-s)x)$$

$$+\left(\nabla f(sy+(1-s)x)\right)^T \underbrace{\left[(ty+(1-t)x)-(sy+(1-s)x)\right]}_{(y-x)(t-s)}$$

$$\rightarrow \quad g(t) \geq g(s) + g'(s)(t-s)$$

which implies that g is convex. therefore f is convex.

- second-order conditions for convexity (of a function):

  assume that f is twice differentiable. then f is convex if and only if $\mathbf{dom}\, f$ is convex and its Hessian is a positive semidefinite matrix, i.e., for all $x \in \mathbf{dom} f$

$$\nabla^2 f(x) \succeq 0$$

- proof of 2nd order cond:

  * first consider the case of $n=1$.

($\Rightarrow$) suppose $f: \mathbb{R} \to \mathbb{R}$ is convex. let $x, y \in \mathbf{dom} f$, $y > x$.

from the first-order condition

$$f(y) \geq f(x) + f'(x)(y-x)$$

$$f(x) \geq f(y) + \underbrace{f'(y)(x-y)}$$

$$-f'(y)(y-x)$$

$$f'(x)(y-x) \le f(y) - f(x) \le f'(y)(y-x)$$

subtract LHS & RHS, then divide by $(y-x)^2$ to get

$$\frac{f'(y) - f'(x)}{y-x} \ge 0$$

taking the limit for $y \to x$ we get $f''(x) \ge 0$.

($\Leftarrow$) suppose $f''(z) \ge 0$ for all $z \in \text{dom } f$. consider two points $x, y \in \text{dom}$, $x < y$. then

$$\begin{aligned} 0 &\le \int_x^y f''(z)(y-z) \\ &= \left(f'(z)(y-z) \right)\Big|_x^y - \int_x^y f'(z)(-1)\, dz \\ &= -f'(x) + f(y) - f(x) \end{aligned}$$

$(y-x)$

aside:

$$\int_x^y (gh)' = \int_x^y gh' + \int_x^y g'h$$

$$\downarrow$$

$$\int_x^y g'h = (gh)\Big|_x^y - \int_x^y gh'$$

here, $f''(z) \sim g'(z)$
$y - z \sim h(z)$

therefor $f(y) \geq f(x) + f'(x)(y-x)$, and f is convex.

* for general n

f is convex if and only if it is convex when restricted to all lines, i.e., iff the function

$$g(t) = f(x_0 + tv)$$

is convex (in t) for all $x_0 \in \mathrm{dom} f$ and all v.

so f is convex iff

$$g''(t) = v^T \left(\nabla^2 f(x_0 + tv) \right) v \geq 0$$

for all $x_0 \in \mathrm{dom} f$, $v \in \mathbb{R}^n$, and t satisfying $x_0 + tv \in \mathrm{dom} f$.

therefore f is convex iff $\nabla^2 f(x) \geq 0$ for all $x \in \mathrm{dom} f$.

- f is strictly convex if

$$\nabla^2 f(x) > 0$$

for all $x \in \text{dom} f$.

- the converse of the above statement is not true:
take $f = x^4$ on $\mathbb{R}$. it is strictly convex but $f''(0) = 0$.

- example: consider the quadratic function $f: \mathbb{R}^n \to \mathbb{R}$

$$f(x) = \frac{1}{2} x^T P x + q^T x + r$$

with $\mathrm{dom} f = \mathbb{R}^n$.

$P \in S^n$
$q \in \mathbb{R}^n$
$r \in \mathbb{R}$

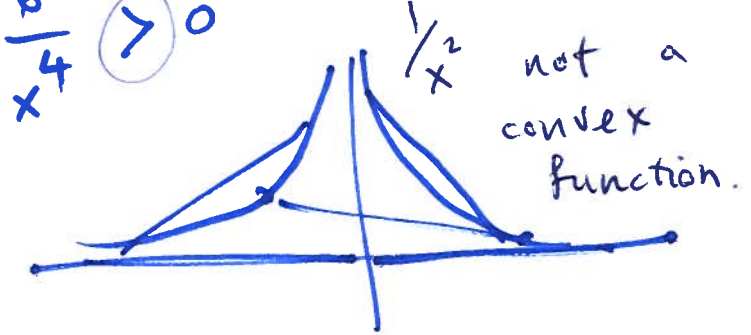
$$\nabla^2 f(x) = P$$

therefor f is convex iff $P \geq 0$

- example: the convexity of $\mathrm{dom} f$ is critical: → for all $x \in \mathrm{dom} f$

$$f(x) = \frac{1}{x^2} \qquad f''(x) = \frac{6}{x^4} > 0$$

$\mathrm{dom} f = \mathbb{R} - \{0\}$ not a convex set.



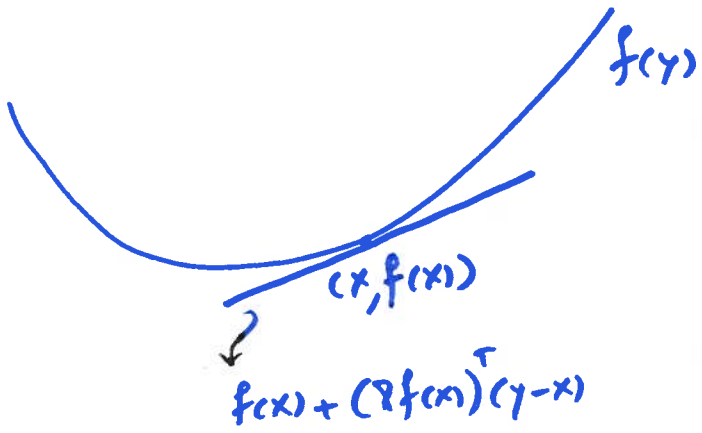


review of last lecture

- 1st-ord. cond. for convexity of a function

$$f(y) \geq f(x) + (\nabla f(x))^T (y - x) \qquad \forall x, y \in \mathbf{dom}\, f,$$

& $\mathbf{dom}\, f$ is convex.



$(\nabla f(x^*) = 0 \rightarrow f(y) \geq f(x^*))$

- 2nd-ord. cond. for convexity of a function

$$\nabla^2 f(x) \succeq 0 \qquad \forall x \in \mathbf{dom}\, f,$$

& $\mathbf{dom}\, f$ is convex.

- convexity of a function can be verified in several ways

  - directly verify

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

  - verify that Hessian is positive semidefinite

$$\nabla^2 f(x) \succeq 0$$

  - restrict function to arbitrary line and verify resulting function (of one variable) is convex.

$$g''(t) \geq 0$$

- examples:

  - negative entropy: $f(x) = x \log x$, $\operatorname{dom} f = \mathbb{R}_{++}$.

$$f'(x) = \log x + 1 \quad , \quad f''(x) = \frac{1}{x}$$

since $f''(x) > 0$ for $x > 0$ then f is (strictly) convex.

- norms: if f is a norm, $f: \mathbb{R}^n \to \mathbb{R}$

$$f(\theta x + (1-\theta) y) \leq f(\theta x) + f((1-\theta) y)$$
$$= \theta f(x) + (1-\theta) f(y)$$

for $0 \leq \theta \leq 1$.

aside: f norm

- $f(x) \geq 0$, $f(x) = 0 \Leftrightarrow x = 0$
- $f(tx) = |t| f(x)$
- $f(x+y) \leq f(x) + f(y)$

- max function: $f(x) = \max_i x_i$

$$f(\theta x + (1-\theta)y) = \max_i (\theta x_i + (1-\theta) y_i)$$

$$\leq \theta \max_i x_i + (1-\theta) \max_i y_i$$

$$= \theta f(x) + (1-\theta) f(y)$$

- quadratic-over-linear function: $f(x,y) = \frac{x^2}{y}, \quad y > 0$

$$\nabla f = \begin{bmatrix} \frac{2x}{y} \\ -\frac{x^2}{y^2} \end{bmatrix}, \quad \nabla^2 f = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix}$$

$$= \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix}^T \begin{bmatrix} y \\ -x \end{bmatrix} \succeq 0 \quad \text{for } y > 0$$

- log-sum-exp: $f(x) = \log(e^{x_1} + \cdots + e^{x_n})$

$$\nabla f(x) = \begin{bmatrix} \frac{e^{x_1}}{e^{x_1} + \cdots + e^{x_n}} \\ \vdots \\ \frac{e^{x_n}}{e^{x_1} + \cdots + e^{x_n}} \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{e^{x_1}(\cdots) - e^{x_1} e^{x_1}}{(\cdots)^2} & \frac{0 - e^{x_2} e^{x_1}}{(\cdots)^2} \\ \frac{0 - e^{x_2} e^{x_1}}{(\cdots)^2} & \frac{e^{x_2}(\cdots) - e^{x_2} e^{x_2}}{(\cdots)^2} \end{bmatrix}$$

$$z = \begin{bmatrix} e^{x_1} \\ \vdots \\ e^{x_n} \end{bmatrix} \qquad \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \operatorname{diag} z = \begin{bmatrix} e^{x_1} & & \\ & \ddots & \\ & & e^{x_n} \end{bmatrix}$$

$$\nabla^2 f(x) = \frac{1}{(\mathbb{1}^T z)^2}\left[(\mathbb{1}^T z)\,\mathrm{diag}\, z - z z^T\right]$$

$$v^T \nabla^2 f(x)\, v = \frac{1}{(\mathbb{1}^T z)^2}\left[\left(\sum_i z_i\right)\left(\sum_i v_i^2 z_i\right) - \left(\sum_i v_i z_i\right)^2\right]$$

from $a_i = v_i\sqrt{z_i}$ $\quad b_i = \sqrt{z_i}$

& Cauchy-Schwarz inequality $\longrightarrow$

aside:

$(a^T a)(b^T b) \geq (a^T b)^2$

we get

$$v^T \nabla^2 f(x)\, v \geq 0 \quad \forall v$$

$$\downarrow$$

$$\nabla^2 f(x) \succeq 0$$

- geometric mean: $f(x) = (x_1 \cdots x_n)^{1/n}$, $\mathbf{dom}\, f = \mathbb{R}^n_{++}$.

$$\frac{\partial^2 f(x)}{\partial x_k^2} = -(n-1)\frac{(x_1 \cdots x_n)^{1/n}}{n^2 x_k^2}, \qquad \frac{\partial^2 f(x)}{\partial x_k \partial x_l} = \frac{(x_1 \cdots x_n)^{1/n}}{n^2 x_k x_l} \quad k \neq l$$

$$q_i = \frac{1}{x_i}$$

$$\nabla^2 f(x) = -\frac{(x_1 \cdots x_n)^{1/n}}{n^2}\left[n\,\mathbf{diag}\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right) - qq^T \right]$$

$$v^T \nabla^2 f(x) v = -\frac{(x_1 \cdots x_n)^{1/n}}{n^2}\left(n \sum_i v_i^2/x_i^2 - \left(\sum_i v_i/x_i\right)^2 \right)$$

let $a_i = 1$ $b_i = v_i/x_i$ and apply Cauchy-Schwarz

$$v^T \nabla f(x) v \leq 0 \quad \longrightarrow \quad \nabla^2 f(x) \preccurlyeq 0$$

8-7

- negative log-determinant: $f(X) = -\log\det X$, $\operatorname{dom} f = S^n_{++}$

consider an arbitrary line $X = X_0 + tV$, $X_0, V \in S^n$.

define $g(t) = f(X_0 + tV)$, $\operatorname{dom} g = \{t \mid X_0 + tV \succ 0\}$

$$g(t) = -\log\det(X_0 + tV) \qquad (\text{assume } X_0 \succ 0)$$

$$= -\log\det\left(X_0^{1/2}(I + tX_0^{-1/2} V X_0^{-1/2}) X_0^{1/2}\right)$$

$$= -\log\det(I + tX_0^{-1/2} V X_0^{-1/2}) - \log\det(\underbrace{X_0^{1/2} X_0^{1/2}}_{X_0})$$

$$= -\log([1+t\lambda_1]\cdots[1+t\lambda_n]) - \log\det X_0$$

$$= -\sum_i \log(1+t\lambda_i) - \log\det X_0$$

$$g'(t) = -\sum_i \frac{\lambda_i}{1+t\lambda_i}, \qquad g''(t) = \sum_i \frac{\lambda_i^2}{(1+t\lambda_i)^2} \geq 0$$

- sublevel set: the α-sublevel set of $f: \mathbb{R}^n \to \mathbb{R}$ is

$$C_\alpha = \{ x \in \mathrm{dom}\, f \mid f(x) \leq \alpha \}$$

sublevel sets of a convex function are convex, for any α.

proof of convexity of C_α: if $x, y \in C_\alpha$, then

$$f(x) \leq \alpha, \quad f(y) \leq \alpha$$

$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$$

$$\leq \theta \alpha + (1-\theta)\alpha = \alpha$$

which means that $\theta x + (1-\theta) y \in C_\alpha$

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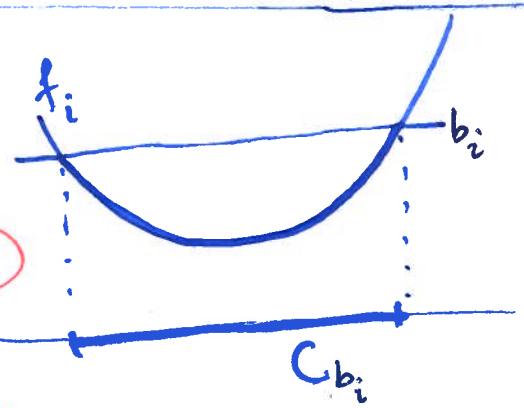
aside: recall first lecture

convex optx. problem

$$\text{min. } f_0(x)$$

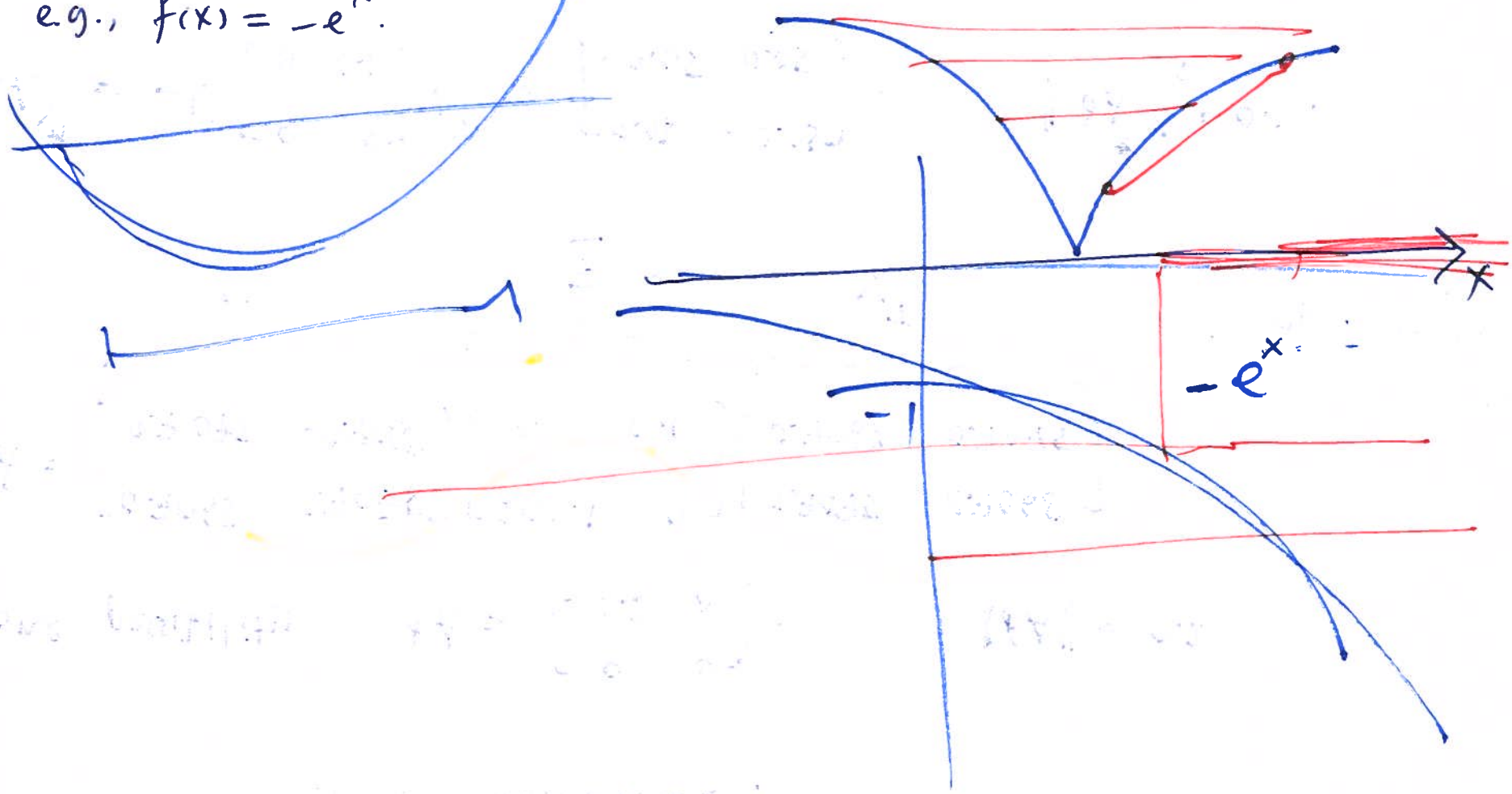
$$\text{s.t. } f_i(x) \leq b_i \quad i = 1, \ldots, m$$

$f_0, f_1, \ldots, f_m$ convex



$$\bigcap_{i=1,\ldots,m} C_{b_i}$$

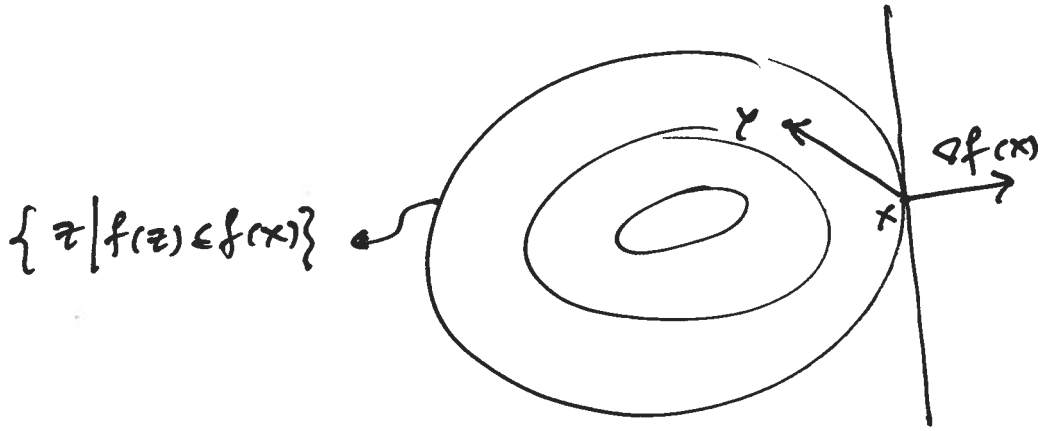
- the converse is <u>not</u> true: a function can have all its sublevel sets convex, but not be a convex function, e.g., $f(x) = -e^x$.



aside: suppose $f: \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable. Then for all $x, y \in \operatorname{dom} f$

$$f(y) \leq f(x) \quad \text{implies} \quad (\nabla f(x))^T (y - x) \leq 0$$

this is actually called the quasi-convexity condition



proof:

$$f(y) \geq f(x) + (\nabla f(x))^T (y - x) \qquad \text{(1st-order convexity cond.)}$$

$$\downarrow$$

$$f(y) - f(x) \geq (\nabla f(x))^T (y - x)$$

$$\downarrow \; 0 \geq f(y) - f(x)$$

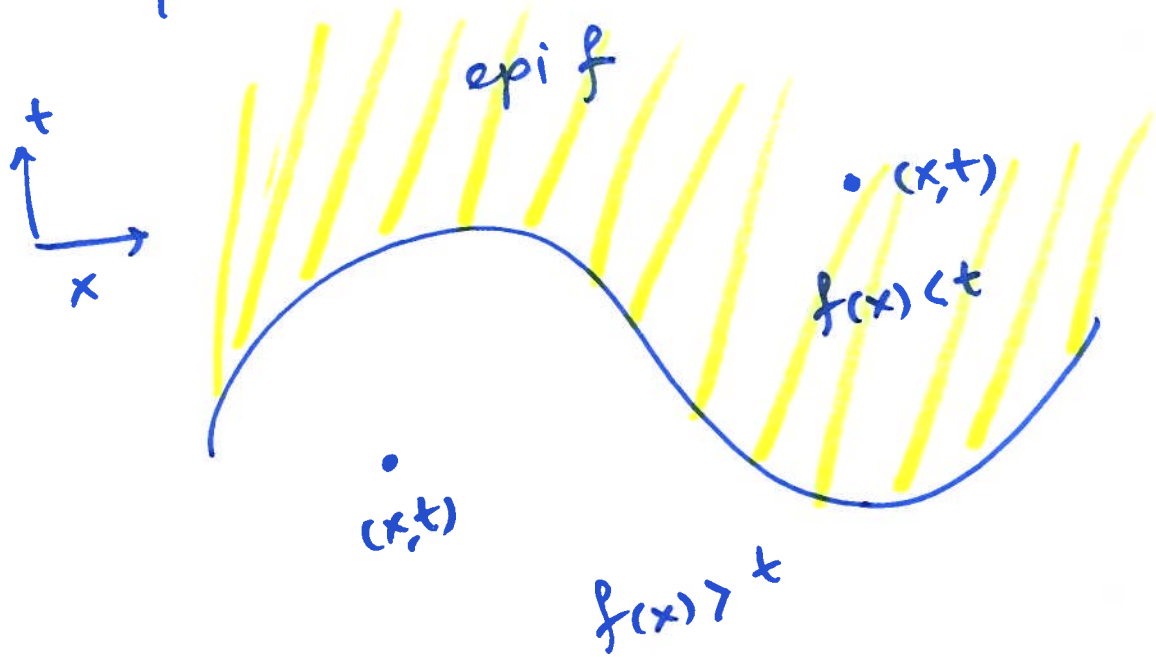
$$(\nabla f(x))^T (y - x) \leq 0$$

- the graph of $f: \mathbb{R}^n \to \mathbb{R}$ is

$$\{(x, f(x)) \mid x \in \operatorname{dom} f\} \quad (\subset \mathbb{R}^{n+1})$$

- epigraph: the epigraph of $f: \mathbb{R}^n \to \mathbb{R}$ is

$$\operatorname{epi} f = \{(x,t) \mid x \in \operatorname{dom} f,\ f(x) \le t\} \quad (\subset \mathbb{R}^{n+1})$$



- a function is convex if and only if its epigraph is a convex set.

link between convex functions & convex sets.

review of last lecture

- convexity of a function can be verified in several ways

  - directly verify

    $$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

    } as in showing convexity of norm

  - verify that Hessian is positive semidefinite

    $$\nabla^2 f(x) \succeq 0$$

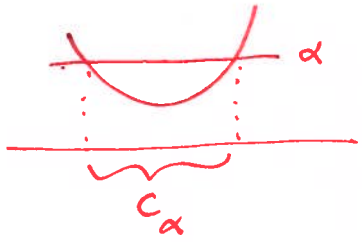
    } as in showing convexity of x^2/y

  - restrict function to arbitrary line and verify resulting function (of one variable) is convex.

    $$g''(t) \geq 0$$

    } as in showing convexity of $-\log\det$

- sublevel set: $C_\alpha = \{ x \mid x \in \mathbf{dom} f,\ f(x) \leq \alpha \}$

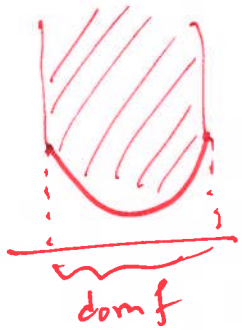


- sublevel sets of a convex function are convex.

  → we use this in the formulation of convex optimization problems

  min. ...
  s.t. $\underbrace{f_i(x)}_{\text{convex function.}} \leq \alpha_i$ } convex set

- epigraph: $\mathbf{epi} f = \{ (x,t) \mid x \in \mathbf{dom} f,\ f(x) \leq t \}$



- a function is convex iff its epigraph is a convex set.

  → this establishes the link between convex funcs & sets

- recall

example: the convex set described by the solution of a linear matrix inequality (LMI) $\{x \mid A(x) \preceq B\}$, with $A(x) = x_1 A_1 + \cdots + x_n A_n$, $A_i, B \in S^m$, can be expressed as the inverse image of the positive semidefinite cone S^m_+ under the affine function $f(x) = B - A(x)$ $(f: \mathbb{R}^n \to S^m)$

$$f^{-1}(S^m_+) = \{x \mid \underbrace{f(x) \in S^m_+}_{B - A(x) \succeq 0}\}$$

aside: $A(x) = x_1 A_1 + \cdots + x_n A_n \preceq B$ $(B, A_i \in S^m)$ is called an LMI in x.

- example: the function $f: \mathbb{R}^n \times S^n \to \mathbb{R}$ defined as

$$f(x, Y) = x^T Y^{-1} x$$

is convex $\operatorname{dom} f = \mathbb{R}^n \times S^n_{++}$. to establish convexity of f thru its epigraph, note that

$$\text{epi } f = \{ (x, Y, t) \mid Y \succ 0,\ x^T Y^{-1} x \le t \}$$

$t - x^T Y^{-1} x \succeq 0$

$$= \left\{ (x, Y, t) \;\middle|\; \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succeq 0,\ Y \succ 0 \right\}$$

an LMI in (x, Y, t) and therefore epi f is a convex set $\to$ f is a convex func.

---

aside: Schur complement

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \iff \begin{cases} A \succ 0 \\ \& \\ S \succeq 0 \end{cases} \quad \text{where} \quad S := C - B^T A^{-1} B$$

also, if $A \succ 0$ then

$$X \succeq 0 \iff S \succeq 0 \qquad \left(\text{i.e., } \begin{cases} X \succeq 0 \\ \& \\ A \succ 0 \end{cases} \iff \begin{cases} S \succeq 0 \\ \& \\ A \succ 0 \end{cases} \right)$$

for the special case of $n=1$ (which gives $f = x^2/y$) the LMI becomes

$$\begin{bmatrix} y & x \\ x & t \end{bmatrix} \succeq 0, \quad y > 0$$

$$\downarrow$$

$$y\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A_1} + x\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_2} + t\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_3} \succeq 0 \quad \text{an LMI}$$

---

aside:

$$\min_{x,y} \ \text{trace}\,(x^T y^{-1} x)$$

$$\downarrow$$

$$\min_{x,y,w} \ \text{trace}\,(W)$$

$$\text{s.t.} \quad W \succeq x^T y^{-1} x \ \rightarrow \ \begin{bmatrix} y & x \\ x^T & W \end{bmatrix} \succeq 0$$

} convex optz.

$x^T y^{-1} x \sim$ scala

$$-\underbrace{(\ )\,|\,|}$$

- geometric interpretation of first-order condition for convexity:

consider the first-order condition for convexity

$$f(y) \geq f(x) + (\nabla f(x))^T (y-x)$$

where f is convex and $x, y \in \mathbf{dom} f$. then, if $(y,t) \in \mathbf{epi} f$

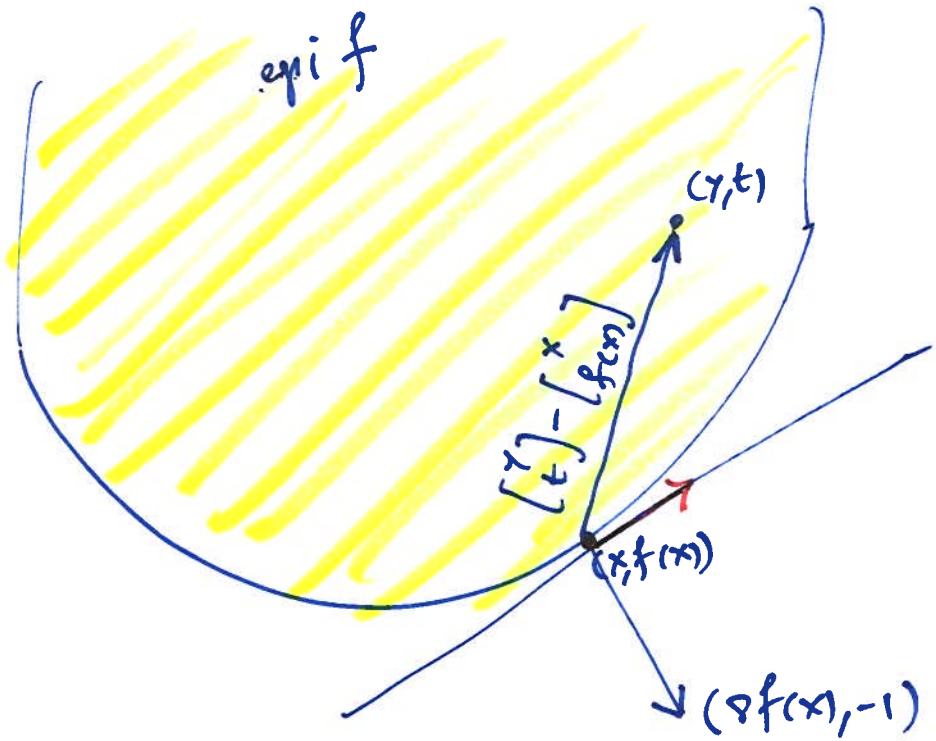
$$t \geq f(y) \geq f(x) + (\nabla f(x))^T (y-x)$$

$$\downarrow$$

$$(\nabla f(x))^T (y-x) - (t - f(x)) \leq 0$$

$$\downarrow$$

$$\begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}^T \left(\begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0$$



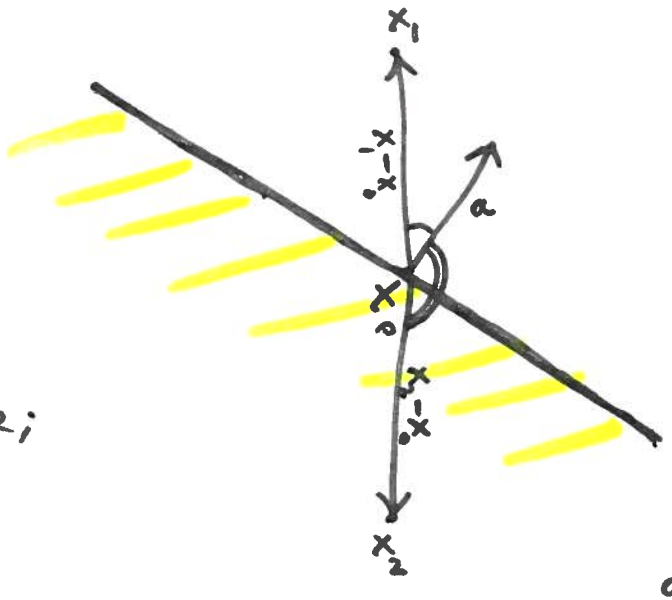
red vector $(\delta x, \delta f)$

$= (\delta x, (\nabla f(x))^T \delta x) \perp (\nabla f(x), -1)$

(we say the hyperplane defined by $(\nabla f(x), -1)$ "supports" epi f at the boundary point $(x, f(x))$).

recall: the shaded set is the half-space determined by

$a^T(x - x_0) \leq 0$



$a^T(x_1 - x_0) > 0$

the vector $x_1 - x_0$ makes an acute angle with a, so x_1 is not in the halfspace;

$a^T(x_2 - x_0) < 0$

the vector $x_2 - x_0$ makes an obtuse angle with a, so x_2 is in the halfspace.

operations that preserve convexity (of functions)

- methods for establishing convexity of function f

① verify directly

e.g., for all $x, y \in \mathbf{dom} f$, $0 \le \theta \le 1$, $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$

or, check $\nabla^2 f(x) \succeq 0$; or restrict function to line & check $g''(t) \ge 0$

② show that f is obtained from other functions by operations that preserve convexity.

- nonnegative weighted sums: if f is convex and $\alpha \ge 0$ then αf is convex. if f_1 and f_2 are both convex then so is $f_1 + f_2$. more generally, a nonnegative weighted sum of convex functions

$$f = w_1 f_1 + \cdots + w_m f_m \qquad \mathbf{dom} f = \bigcap_i \mathbf{dom} f_i$$

is convex

a-7

- composition with affine mapping: suppose $f: \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$, and $b \in \mathbb{R}^n$. define $g: \mathbb{R}^m \to \mathbb{R}$ by

$$g(x) = f(Ax+b)$$

with $\mathbf{dom}\, g = \{x \mid Ax+b \in \mathbf{dom}\, f\}$. then if f is convex, so is g.

- pointwise maximum and supremum: if f_1 and f_2 are convex functions, then their pointwise maximum f, defined by

$$f(x) = \max\{f_1(x), f_2(x)\}$$



with $\mathbf{dom}\, f = \mathbf{dom}\, f_1 \cap \mathbf{dom}\, f_2$, is also convex.

to see this, if $0 \le \theta \le 1$ and $x, y \in \mathbf{dom}\, f$, then

$$\begin{aligned} f(\theta x + (1-\theta)y) &= \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\ &\le \max\{\theta f_1(x) + (1-\theta) f_1(y), \theta f_2(x) + (1-\theta) f_2(y)\} \\ &\le \max\{\theta f_1(x), \theta f_2(x)\} + \max\{(1-\theta) f_1(y), (1-\theta) f_2(y)\} \\ &= \theta \max\{f_1(x), f_2(x)\} + (1-\theta)\max\{f_1(y), f_2(y)\} \\ &= \theta f(x) + (1-\theta) f(y) \end{aligned}$$

which means that f is convex.

aside:

$\max\{f_1+g_1, f_2+g_2\}$
$\le \max\{f_1, f_2\}$
$+ \max\{g_1, g_2\}$

- more generally, if $f_1, \ldots, f_m$ are convex, then so is

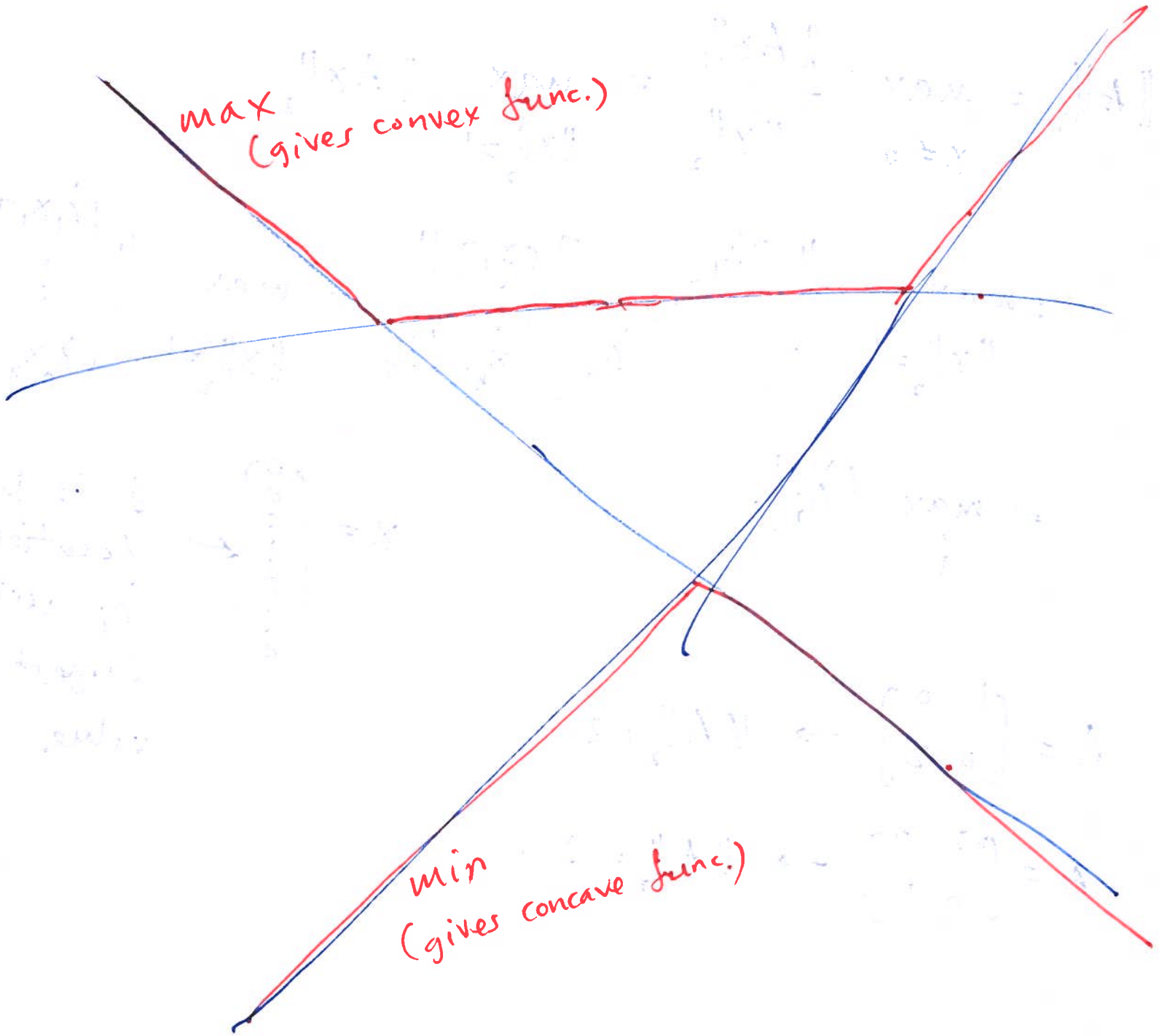
$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}.$$

- example: the function

$$f(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$$

defines a piecewise-linear function, and is convex.

conversely, any piecewise-linear convex function with L or fewer regions can be expressed as above.


max
(gives convex func.)
min
(gives concave func.)

- example: for $x \in \mathbb{R}^n$, we denote by $x_{[i]}$ the i^{th} largest component of x, i.e.,

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}.$$

then the function f defined as

$$f(x) = \sum_{i=1}^{r} x_{[i]}$$

i.e., the sum of the r largest elements of x, is a convex function. to see this, note that

$$f(x) = \max \left\{ x_{i_1} + \cdots + x_{i_r} \;\middle|\; \begin{array}{l} i_1, \ldots, i_r \text{ are } r \text{ distinct} \\ \text{integer values between 1\&n} \end{array} \right\}$$

$$= \max \left\{ x_{i_1} + \cdots + x_{i_r} \;\middle|\; 1 \leq i_1 < \cdots < i_r \leq n \right\}$$

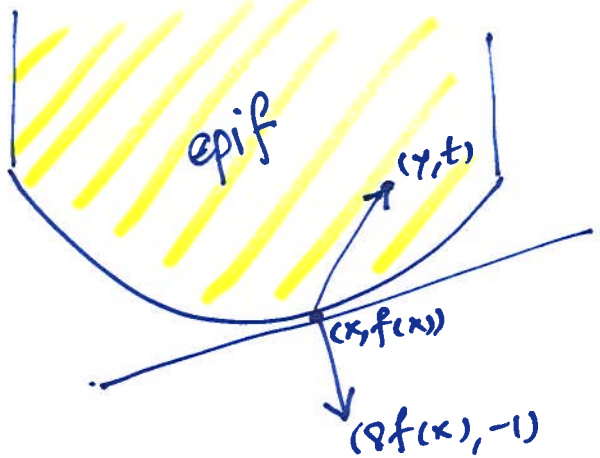
(this is the pointwise maximum of $\binom{n}{r}$ linear functions

review of last lecture

- geometric interp. of 1st ord. cond. for convexity in terms of epif.

$$f(y) \geq f(x) + (\nabla f(x))^T (y - x)$$

$$\Leftrightarrow (y, t) \in \text{epi} f$$



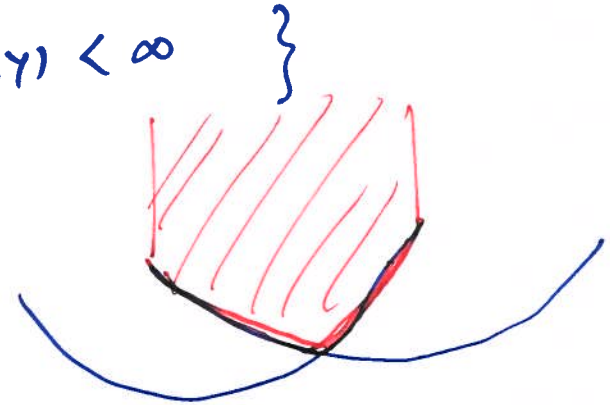
- operations that preserve convexity of functions
  - nonnegative weighted sums
  - composition with affine mapping
  - pointwise maximum/supremum
  - ⋮ } more today

- the pointwise maximum property extends to pointwise supremum over an infinite set of convex functions. if for each $y \in \mathcal{A}$, $f(x,y)$ is convex in x, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x,y)$$

is convex in x, with

$$\operatorname{dom} g = \{ x \mid (x,y) \in \operatorname{dom} f \text{ for all } y \in \mathcal{A}, \sup_{y \in \mathcal{A}} f(x,y) < \infty \}$$



- in terms of epigraphs, the pointwise supremum of functions corresponds to the intersection of epigraphs

$$\operatorname{epi} g = \bigcap_{y \in \mathcal{A}} \operatorname{epi} f(\cdot, y).$$

convexity follows from fact that intersection of family of convex sets is convex.

• example: Let $C \subseteq \mathbb{R}^n$ with $C \neq \emptyset$. the support function S_C associated with C is defined as

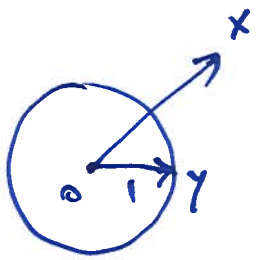
$$S_C(x) = \sup\{x^T y \mid y \in C\}$$

with $\operatorname{dom} S_C = \{x \mid \sup_{y \in C} x^T y < \infty\}$.

for each $y \in C$, $y^T x$ is a linear function of x, hence $S_C(x)$ is convex.

for instance, if $C = B$, (unit Euclidean ball)

then $S_C(x) = \|x\|_2$, since



$$y^* = \frac{x}{\|x\|_2}$$

$$x^T y^* = x^T \frac{x}{\|x\|_2} = \frac{\|x\|_2^2}{\|x\|_2} = \|x\|_2 .$$

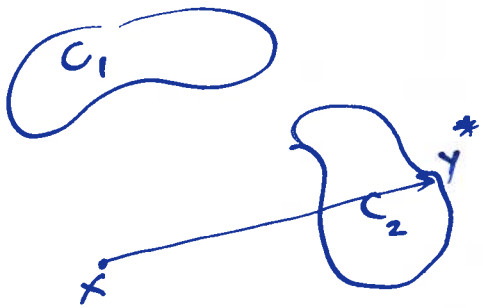
- example: let $C \subseteq \mathbb{R}^n$. the distance (in any norm) to the farthest point of C

$$f(x) = \sup_{y \in C} \|x - y\|$$

is convex.

this is because for any y the function $\|x-y\|$ is convex in x.
since f is the pointwise supremum of a family of convex functions indexed by y, it is convex in x.

$C = C_1 \cup C_2$



- example: we will see that the pointwise maximum property is critical in proving that the "dual" of an optimization problem is convex (regardless of whether original problem is convex)

- example: the function $f(X) = \lambda_{max}(X)$, with $\mathrm{dom} f = S^n$, is convex. note that f can be written as

$$f(X) = \sup \{ y^T X y \mid \|y\|_2 = 1 \}$$

& thus f is the pointwise supremum of a family of linear functions of X indexed by y.

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aside:

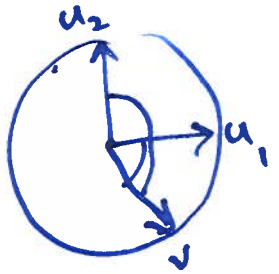
$$X = U \Lambda U^T = \sum_i \lambda_i u_i u_i^T$$

$$U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$u_i^T u_j = 0 \quad i \neq j$

$\|u_i\|_2 = 1 \quad \forall i$

$$v^T X v = \sum_i \lambda_i (v^T u_i)(u_i^T v) = \sum_i \lambda_i |v^T u_i|^2 \qquad \|v\|_2 = 1$$



$$v^T u_i = \underbrace{\|v\|}_{1} \underbrace{\|u_i\|}_{1} \cos\theta$$

in 2-dim ↳

$$\lambda_1 (\cos\theta_1)^2 + \lambda_2 \underbrace{(\cos\theta_2)^2}_{\sin\theta_1} \leq \lambda_1 \qquad \lambda_1 \geq \lambda_2$$

will get λ_1 if $\theta_1 = 0$ & thus $v = u_1$

$$\cdots \leq \lambda_2 \qquad \lambda_2 \geq \lambda_1$$

will get λ_2 if $\theta_2 = 0$ & thus $v = u_2$

↓

$$\sup v^T X v = \max_i \{\lambda_i\}.$$

10-6

- example: the induced 2-norm $f(X) = \|X\|_2$ with $\mathrm{dom} f = \mathbb{R}^{p\times q}$ is convex. ($\|X\|_2$ is also equal to the largest singular value of X.)

$$\|X\|_2 = \sup_{y\neq 0} \frac{\|Xy\|_2}{\|y\|_2}$$

$$= \sup_{\|v\|_2=1} \|Xv\|_2 \qquad \left(v = \frac{y}{\|y\|_2} \rightarrow \|v\|_2 = 1\right)$$

$$= \sup_{\substack{\|v\|_2=1 \\ \|u\|_2=1}} u^T(Xv) \qquad \left(\|z\|_2 = \sup_{\|w\|_2=1} w^T z \text{ with sup achieved for } w^* = \frac{z}{\|z\|_2}\right)$$

$$f(X) = \sup\{u^T X v \mid \|u\|_2 = 1, \|v\|_2 = 1\}$$

so f can be expressed as the pointwise supremum of a family of linear functions of X.

aside: this generalizes to the $\|\cdot\|_{a,b}$ induced norm

$$\|X\|_{a,b} = \sup_{y \neq 0} \frac{\|Xy\|_a}{\|y\|_b}$$

$$= \sup_{\|v\|_b = 1} \|Xv\|_a$$

$$= \sup_{\substack{\|u\|_{a^*}=1 \\ \|v\|_b=1}} u^T(Xv)$$

($\|\cdot\|_{a^*}$ is dual norm of $\|\cdot\|_a$)

$$\|X\|_{a,b} = \sup \{ u^T X v \mid \|u\|_{a^*}=1, \ \|v\|_b = 1 \}$$

since $\|X\|_{a,b}$ can be expressed as the supremum of linear functions of X then it is convex.

- minimization: if f is convex in (x,y) and C is a convex nonempty set, then

$$g(x) = \inf_{y \in C} f(x,y)$$

is convex in x provided that $g(x) > -\infty$ for all x, and

$$\operatorname{dom} g = \{ x \mid (x,y) \in \operatorname{dom} f \text{ for some } y \in C \}$$

to prove this, for $x_1, x_2 \in \operatorname{dom} g$ let $y_1, y_2 \in C$ achieve the inf above,

$$g(x_1) = f(x_1, y_1)$$
$$g(x_2) = f(x_2, y_2)$$

then

↓

$$g(\theta x_1 + (1-\theta)x_2)$$

$$= \inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y)$$

$$\leq f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)$$

$$\leq \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2)$$

$$= \theta g(x_1) + (1-\theta) g(x_2)$$

which means that g is convex.

$$f\left(\begin{bmatrix} \theta x_1 \\ \theta y_1 \end{bmatrix} + \begin{bmatrix} (1-\theta)x_2 \\ (1-\theta)y_2 \end{bmatrix}\right) = f\left(\theta \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$\leq \theta f\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + (1-\theta) f\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

- in terms of epigraphs, with f, g, and C defined as earlier and assuming the inf over $y \in C$ is attained for each x, we have

$$\text{epi } g = \{(x,t) \mid (x,y,t) \in \text{epi } f \text{ for some } y \in C\}$$

epi g is convex because it is the projection of epi f on some of its components.

epi f: a convex set, since f is convex function

- example: suppose the quadratic function

$$f(x,y) = x^T A x + 2x^T B y + y^T C y$$

(where A and C are symmetric) is convex in (x,y), and $C \succ 0$. then from the convexity of f it follows that $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0$, since

$$f(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \underbrace{\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}}_{P} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{is convex iff} \quad P \succeq 0.$$

then the solution of $\underset{y}{\text{minimize}}\ f(x,y)$ with $C \succ 0$ is given by $y^* = -C^{-1}B^T x$.

(obtained from setting $\nabla_y f = 2B^T x + 2Cy = 0$.)

we have

$$g(x) = \inf_y f(x,y) = f(x, y^*(x)) = x^T \underbrace{(A - BC^{-1}B^T)}_{S} x$$

from the minimization rule we know that g is convex, so we conclude that $A - BC^{-1}B^T \succeq 0$.

(we have thus proven that if $C \succ 0$, then

$$P \succeq 0 \iff S \succeq 0.)$$

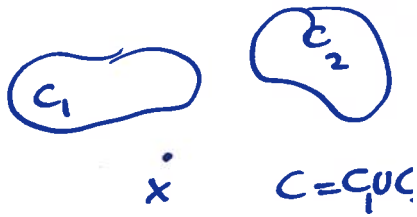
(recall that $S = A - BC^{-1}B^T$ is called the Schur complement of C in the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$.)

- example: suppose h is convex. then $g(x) = \inf \{ h(y) \mid Ay = x \}$ is convex. to see this, define $f(x,y) = \begin{cases} h(y) & \text{if } Ay = x \\ \infty & \text{else} \end{cases}$ which is convex in (x,y). then g is the min of f over y and thus convex.

- example: the distance of a point x to a set $S \subseteq \mathbb{R}^n$, in any norm, is defined as

$$\text{dist}(x,S) = \inf_{y \in S} \|x-y\|.$$

the function $\|x-y\|$ is convex in (x,y), so if the set S is convex then $\text{dist}(x,S)$ is a convex function of x.



- to summarize:

$g(x) = \sup_{y \in \mathcal{A}} f(x,y)$ is convex in x if $f(x,y)$ is convex in x for each $y \in \mathcal{A}$.

$g(x) = \inf_{y \in C} f(x,y)$ is convex in x if $f(x,y)$ is convex in (x,y) and C is convex.

- composition (in general, not just with affine mapping):

  we examine conditions on $g: \mathbb{R}^n \to \mathbb{R}^k$ and $h: \mathbb{R}^k \to \mathbb{R}$ that guarantee convexity of their composition

  $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$ defined by

  $$f(x) = h(g(x)) \qquad \mathbf{dom} f = \{ x \mid x \in \mathbf{dom} g,\ g(x) \in \mathbf{dom} h \}$$

  we restrict ourselves to $n=1$ (since convexity is determined by the behavior of a function on arbitrary lines that intersect its domain).

  assume g and h are twice differentiable and $\mathbf{dom} g = \mathbb{R}$ and $\mathbf{dom} h = \mathbb{R}^k$.

  the convexity of f reduces to $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

$k=1$

$$f(x) = h(g(x))$$

$$f''(x) = h''(g(x))\left(g'(x)\right)^2 + h'(g(x))\, g''(x)$$

① f is convex if h is convex and nondecreasing, and g is convex.

② f is convex if h is convex and nonincreasing, and g is concave.

③ f is concave if h is concave and nondecreasing, and g is concave

④ f is concave if h is concave and nonincreasing, and g is convex

- examples:
  - if g is convex then $e^{g(x)}$ is convex ($h(x)=e^x$; $h', h'' \geq 0$, $g'' \geq 0 \rightarrow$ ①)
  - if g is concave (& positive) $\log(g(x))$ is concave ($h(x)=\log x$; $h' \geq 0$, $h'' \leq 0$, $g'' \leq 0 \rightarrow$ ③)
  - if g is concave (& positive) $1/g(x)$ is convex ($h(x)=1/x$; $h' \leq 0$, $h'' \geq 0$, $g'' \leq 0 \rightarrow$ ②)
  - if g is convex (& nonnegative) & $p \geq 1$, $(g(x))^p$ is convex ($h(x)=x^p$; $h', h'' \geq 0$, $g'' \geq 0 \rightarrow$ ①)
  - if g is convex (& negative) $-\log(-g(x))$ is convex ($h(x)=-\log(-x)$; $h', h'' \geq 0$, $g'' \geq 0 \rightarrow$ ①)

$k \geq 1$ $\qquad f(x) = h(g(x)) = h(g_1(x), \cdots, g_k(x)) \qquad g_i : \mathbb{R} \to \mathbb{R}$

$$f''(x) = \left(g'(x)\right)^T \nabla^2 h(g(x)) \left(g'(x)\right) + \left(\nabla h(g(x))\right)^T g''(x)$$

① f is convex if h is convex and nondecreasing in each argument, and g_i are convex

② f is convex if h is convex and nonincreasing in each argument, and g_i are concave

③ f is concave if h is concave and nondecreasing in each argument, and g_i are concave

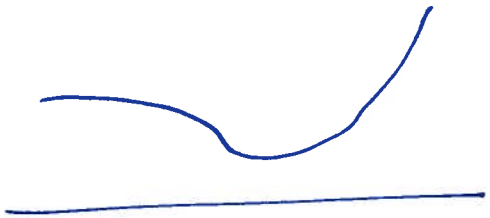
④ f is concave if h is concave and nonincreasing in each argument, and g_i are convex.

review of last lecture

- operations that preserve convexity of functions
  - nonnegative weighted sums
  - composition with affine mapping
  - pointwise maximum/supremum
  - minimization
  - general composition

- quasiconvex functions: a function $f: \mathbb{R}^n \to \mathbb{R}$ is quasiconvex (or unimodal) if its domain & all its sublevel sets

$$S_\alpha = \{ x \in \operatorname{dom} f \mid f(x) \leq \alpha \},$$



for $\alpha \in \mathbb{R}$, are convex

(compare with condition that a function is convex iff its epigraph is convex)

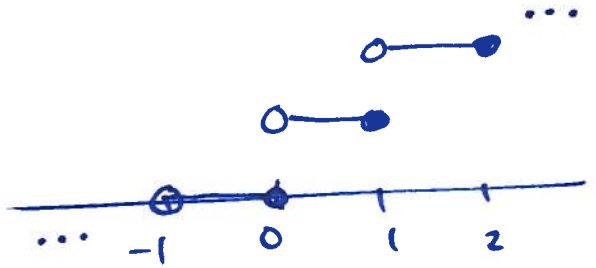
a function is quasiconcave if $-f$ is quasiconvex, i.e., every superlevel set $\{ x \mid f(x) \geq \alpha \}$ is convex

a function is quasilinear if it is both quasiconvex & quasiconcave.
if a function is quasilinear then its domain and every level set $\{ x \mid f(x) = \alpha \}$ is convex.

quasiconvex functions can be concave or discontinuous.

convex functions have convex sublevel sets & so are quasiconvex.

- examples:
  - logarithm: $\log x$ on $\mathbb{R}_{++}$ is quasiconvex (and quasiconcave, hence quasilinear)
  - ceiling function: $\text{ceil}(x)$ is quasiconvex (and quasiconcave, hence quasilinear)



  - we define the length of $x \in \mathbb{R}^n$ as the largest index of a nonzero component,

$$f(x) = \max \{ i \mid x_i \neq 0 \}$$

  - this function is quasiconvex on $\mathbb{R}^n$ since its sublevel sets $f(x) \leq \alpha$ are subspaces

$$f(x) \leq \alpha \iff x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\lfloor \alpha \rfloor} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow x_{\lfloor \alpha \rfloor + 1} = 0$$

- example: consider $f: \mathbb{R}^2 \to \mathbb{R}$ with $\operatorname{dom} f = \mathbb{R}^2_+$ and

$$f(x_1, x_2) = x_1 x_2.$$

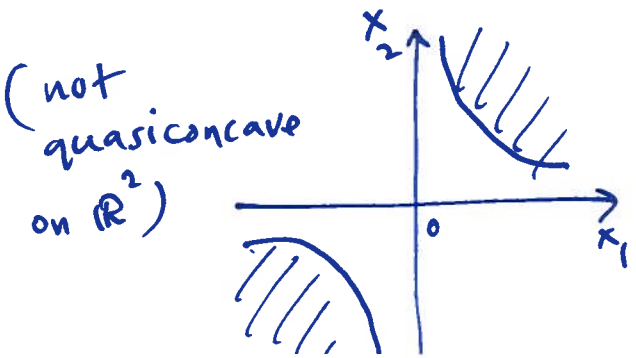
this function is neither convex nor concave since

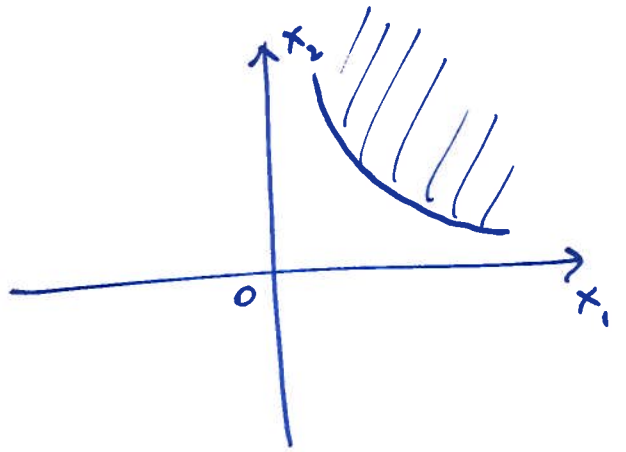
$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is indefinite (it has one positive and one negative eigenvalue). f is quasiconcave because the superlevel sets

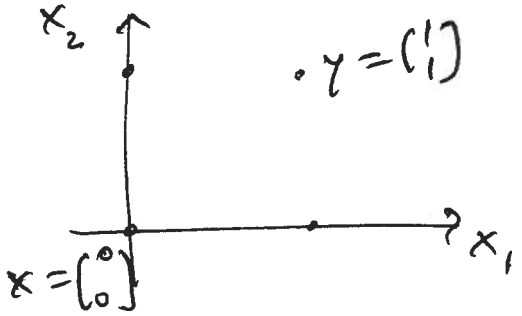
$$\{ x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq \alpha \}$$

are convex for all α.





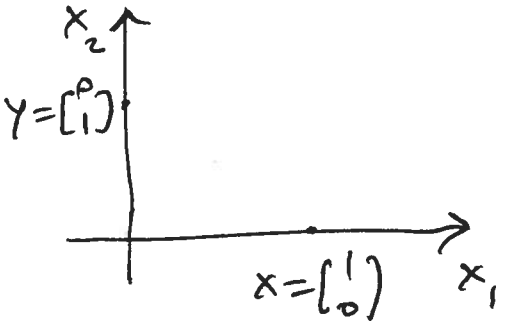
aside: $f(x) = x_1 x_2$



① $f(\theta x + (1-\theta)y) = f\left(\begin{bmatrix} 1-\theta \\ 1-\theta \end{bmatrix}\right) = (1-\theta)^2$

$\theta f(x) + (1-\theta) f(y) = \theta \cdot 0 + (1-\theta) \cdot 1 = 1-\theta$

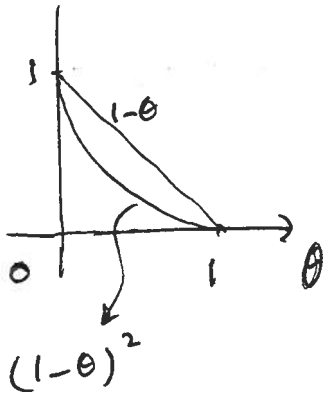
$f(x) = x_1 x_2$



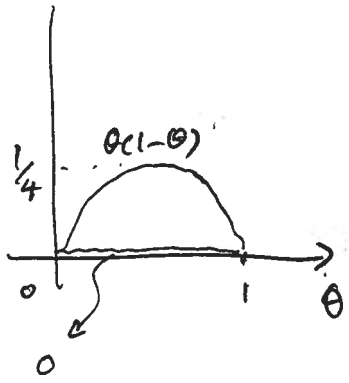
② $f(\theta x + (1-\theta)y) = f\left(\begin{bmatrix} \theta \\ 1-\theta \end{bmatrix}\right) = \theta(1-\theta)$

$\theta f(x) + (1-\theta) f(y) = \theta \cdot 0 + (1-\theta) \cdot 0 = 0$

① →

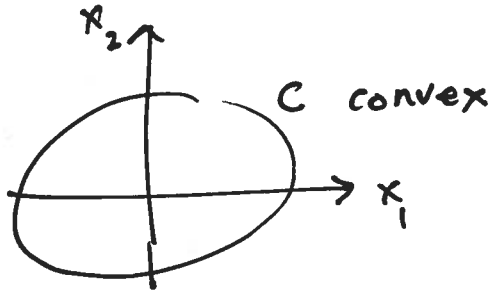


② →



aside:

max. $x_1 x_2$ (not a convex problem)
s.t. $x \in C$
$x \succ 0$



$\updownarrow$

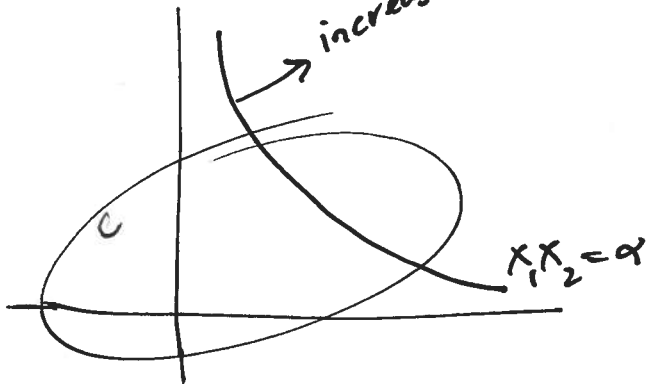
max. α^2
s.t. $\alpha^2 = x_1 x_2 \iff x_1 x_2 \geq \alpha^2 \overset{x \succ 0}{\iff} x_1 - \frac{\alpha^2}{x_2} \geq 0 \iff \begin{bmatrix} x_1 & \alpha \\ \alpha & x_2 \end{bmatrix} \succeq 0$
$x \in C \iff f(x) \leq 0$
$x \succ 0$ $\quad f$: convex

$\updownarrow$

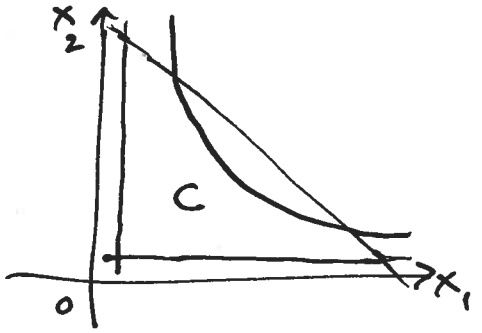
max. α
s.t. $\begin{bmatrix} x_1 & \alpha \\ \alpha & x_2 \end{bmatrix} \succeq 0$
$f(x) \leq 0$
$x \succ 0$
$\alpha > 0$

can replace objective α^2 with α.
(we'll see full justification of this later)

(convex problem!)



(set above $x_1 x_2 = \alpha$ curve is convex due to quasiconcavity)



aside:

$$\max. \quad x_1 x_2$$
$$\text{s.t.} \quad x_1 \geq \varepsilon_1, \; x_2 \geq \varepsilon_2$$
$$x_1 + x_2 \leq \alpha, \; x_1^{\beta_1} x_2^{\beta_2} \leq \mu$$

$\updownarrow$ (change of var.)

$$\max. \quad e^{z_1} e^{z_2}$$
$$\text{s.t.} \quad e^{z_1} \geq \varepsilon_1, \; e^{z_2} \geq \varepsilon_2$$
$$e^{z_1} + e^{z_2} \leq \alpha, \; e^{\beta_1 z_1} e^{\beta_2 z_2} \leq \mu$$

$\updownarrow$ (transformation of objective & constraint functions)

$$\max. \quad \log(e^{z_1} e^{z_2})$$
$$\text{s.t.} \quad \log\left(\frac{\varepsilon_1}{e^{z_1}}\right) \leq 0, \; \log\left(\frac{\varepsilon_2}{e^{z_2}}\right) \leq 0 \qquad \left(\frac{\varepsilon_i}{e^{z_i}} \leq 1 \iff \log \frac{\varepsilon_i}{e^{z_i}} \leq 0\right)$$
$$\log\left(\frac{e^{z_1} + e^{z_2}}{\alpha}\right) \leq 0, \; \log\left(\frac{e^{\beta_1 z_1} e^{\beta_2 z_2}}{\mu}\right) \leq 0$$

$\updownarrow$

$$\max. \quad z_1 + z_2$$
$$\text{s.t.} \quad \delta_1 - z_1 \leq 0, \; \delta_2 - z_2 \leq 0$$
$$\underbrace{\log(e^{z_1} + e^{z_2})}_{\text{"log-sum-exp" convex function}} \leq \log \alpha, \; \beta_1 z_1 + \beta_2 z_2 - \eta \leq 0$$

$\left(\varepsilon_1 = e^{\delta_1}, \; \varepsilon_2 = e^{\delta_2}, \; \mu = e^{\eta}\right)$

convex problem!

- example: the function

$$f(x) = \frac{a^T x + b}{c^T x + d}$$

with $\mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$ is quasiconvex and quasiconcave, hence quasilinear.

$$S_\alpha = \{x \mid c^T x + d > 0,\ (a^T x + b)/(c^T x + d) \le \alpha\}$$

$$= \{x \mid c^T x + d > 0,\ a^T x + b - \alpha(c^T x + d) \le 0\}$$

which is convex because it is the intersection of an open halfspace and a closed halfspace.

aside:

$$\min. \quad \frac{a^T x + b}{c^T x + d} \qquad \text{(domain of objective} = \{x \mid c^T x + d > 0\}\text{)}$$

$$\text{s.t.} \quad Gx \preccurlyeq h$$
$$Ax = f$$

← quasilinear objective

$$\updownarrow$$

$$\min. \quad a^T y + bz$$
$$\text{s.t.} \quad Gy \preccurlyeq hz$$
$$Ay = fz$$
$$e^T y + dz = 1$$
$$z \geq 0$$

which is a linear program & convex;

once solution (y^*, z^*) is found, x^* is obtained from $x^* = \frac{y^*}{z^*}$

(the trick lies in the transformation $y = \frac{x}{c^T x + d}$, $z = \frac{1}{c^T x + d}$).

- example: the function

$$f(x) = \frac{\|x-a\|_2}{\|x-b\|_2}$$

for $a, b \in \mathbb{R}^n$, is quasiconvex on the halfspace $\{x \mid \|x-a\|_2 \le \|x-b\|_2\}$ (where $f(x) \le 1$). to see this, consider the α-sublevel set of f, with $\alpha \ge 0$

$$\|x-a\|_2 \le \alpha \|x-b\|_2$$

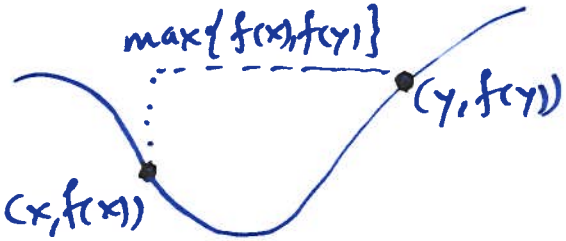
$$\downarrow$$

$$(1-\alpha^2)x^Tx - 2(a-\alpha^2 b)^T x + a^Ta - \alpha^2 b^T b \le 0$$

which describes a convex set (in fact, a Euclidean ball) if $\alpha \le 1$.

- a function f is quasiconvex if and only if $\mathbf{dom} f$ is convex and for any $x, y \in \mathbf{dom} f$ and $0 \le \theta \le 1$

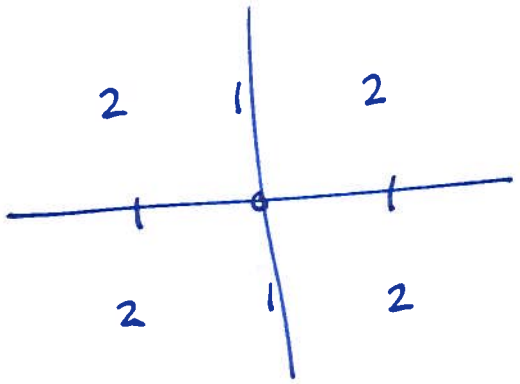
$$f(\theta x + (1-\theta)y) \le \max\{f(x), f(y)\}$$



(compare with $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$ for convex func.)

- example: the cardinality $\text{card}(x)$ of a vector $x \in \mathbb{R}^n$ is its number of nonzero components. this function is quasiconcave on $\mathbb{R}^n_+$.

(not quasiconcave on $\mathbb{R}^n$.)



to see this, we use that for $x, y \geq 0$

$$\text{card}(x+y) \geq \min\{\text{card}(x), \text{card}(y)\}$$

and set $f(x) = -\text{card}(x)$. then for $0 \leq \theta \leq 1$

$$\begin{aligned} f(\theta x + (1-\theta)y) &= -\text{card}(\theta x + (1-\theta)y) \\ &\leq \max\{-\text{card}(\theta x), -\text{card}((1-\theta)y)\} \\ &\leq \max\{-\text{card}(x), -\text{card}(y)\} \\ &= \max\{f(x), f(y)\} \end{aligned}$$

$\rightarrow$ -card quasiconvex

$\rightarrow$ card quasiconcave

- example: the function $\text{rank}(X)$ is quasiconcave on S_+^n.

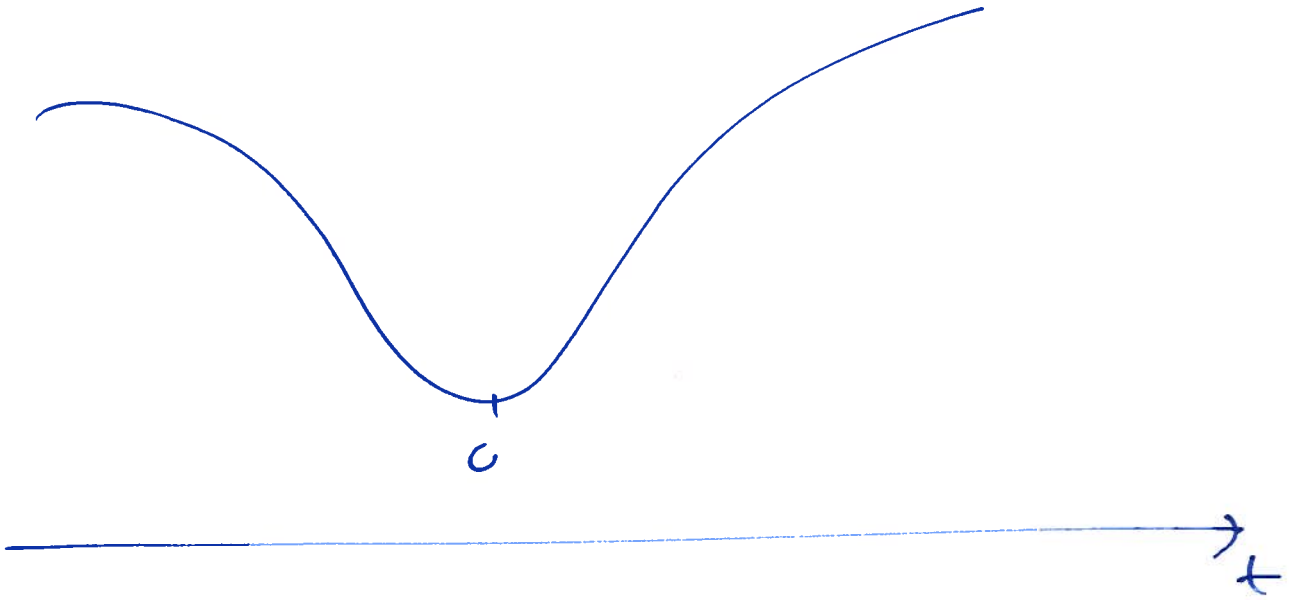
  this follows from fact that for $X, Y \in S_+^n$ we have

  $$\text{rank}(X+Y) \geq \min\{\text{rank}(X), \text{rank}(Y)\}.$$

- f is quasiconvex if and only if its restriction to any line intersecting its domain is quasiconvex. (resulting in function on $\mathbb{R}$)

- a continuous function $f: \mathbb{R} \to \mathbb{R}$ is quasiconvex if and only if at least one of the following conditions holds

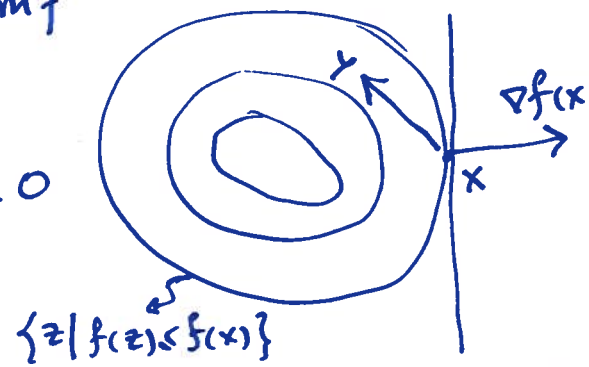
  * f is nondecreasing
  * f is nonincreasing
  * there is a point $c \in \text{dom} f$ such that for $t \leq c$ $(t \in \text{dom} f)$ f is nondecreasing, and for $t \geq c$ $(t \in \text{dom} f)$ f is nonincreasing



the point c can be chosen as any point which is a global minimizer of f.

- first-order conditions for quasiconvexity of functions: suppose $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then f is quasiconvex iff $\mathbf{dom} f$ is convex & for all $x, y \in \mathbf{dom} f$

$$f(y) \leq f(x) \implies (\nabla f(x))^T (y - x) \leq 0$$



(compare with $f(y) \geq f(x) + (\nabla f(x))^T (y - x)$ for convex functions)

- for convex functions, $\nabla f(x) = 0$ implies global optimality of x; this conclusion can not be made for quasiconvex functions.

$$\underbrace{0 \geq f(y) - f(x)}_{\text{for convex functions}} \geq (\nabla f(x))^T (y - x)$$

for convex functions → convex functions satisfy 1st-ord quasiconvexity condition.

- if $f(x,y)$ is quasiconvex in x for each y, then the function

$$g(x) = \sup_{y \in A} f(x,y)$$

is quasiconvex

- if $f(x,y)$ is quasiconvex (jointly) in x and y and C is a convex set, then the function

$$g(x) = \inf_{y \in C} f(x,y)$$

is quasiconvex.

review of last lecture

- quasiconvex functions: a function is quasiconvex if and only if
  - all its sublevel sets $\underbrace{\bigwedge}_{(\forall \alpha)} S_\alpha = \{x \mid x \in \mathbf{dom} f,\ f(x) \le \alpha\}$ are convex.
  - for any $x, y \in \mathbf{dom} f$ and $0 \le \theta \le 1$, $f(\theta x + (1-\theta)y) \le \max\{f(x), f(y)\}$.
  - its restriction to any line is quasiconvex on $\mathbb{R}$.
  - f is nondecreasing, or, it is nonincreasing, or, there is a point c such that for $t \le c$ f is nonincreasing and for $t \ge c$ f is nondecreasing.
  - for all $x, y \in \mathbf{dom} f$ $\quad f(y) \le f(x) \Rightarrow (\nabla f(x))^T (y-x) \le 0$

(in all the above conditions, it is assumed that <u>dom$f$ is convex</u>)

- if $f(x,y)$ is quasiconvex in x for each y, then the function

$$g(x) = \sup_{y \in A} f(x,y)$$

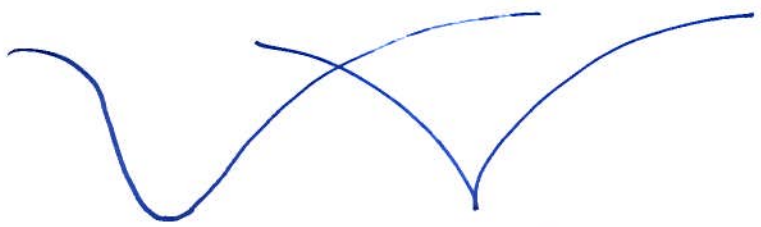
is quasiconvex

- if $f(x,y)$ is quasiconvex (jointly) in x and y and C is a convex set, then the function

$$g(x) = \inf_{y \in C} f(x,y)$$

is quasiconvex.

- the sum of two quasiconvex functions is not necessarily quasiconvex



- representation of quasiconvex functions via family of convex functions: it is useful to represent sublevel sets (convex) of a quasiconvex function via inequalities involving convex functions. we seek convex functions $\varphi_t : \mathbb{R}^n \to \mathbb{R}$, indexed by $t \in \mathbb{R}$, such that

$$f(x) \le t \iff \varphi_t(x) \le 0$$

φ_t must also satisfy the property that for all $x \in \mathbb{R}^n$, $\varphi_t(x) \le 0 \Rightarrow \varphi_s(x) \le 0$ for $s \ge t$.

this is satisfied if for each x, $\varphi_t(x)$ is a nonincreasing function of t, i.e.,

$$\varphi_s(x) \le \varphi_t(x) \quad \text{when } s \ge t.$$

aside:

let $s \ge t$ & fix x

$f(x) \le t \iff \varphi_t(x) \le 0$

$f(x) \le s \iff \varphi_s(x) \le 0$

aside: it can be shown that such a family of functions

- note that for a given f, ϕ_t is not unique. we are usually interested in a family ϕ_t with nice properties, such as differentiability.

- example: suppose p is a convex function, q is a concave function, and $p(x) \geq 0$ & $q(x) > 0$ on a convex set C. then $f(x) = p(x)/q(x)$ on C is quasiconvex. we have

$$f(x) \leq t \iff \underbrace{p(x) - tq(x)}_{\phi_t(x)} \leq 0 \quad \text{for} \quad t \geq 0$$

for each t, ϕ_t is convex in x, and for each x, ϕ_t is decreasing in t.

aside: to see why this representation is useful, consider

minimize $f_0(x)$ (quasiconvex)
subject to $f_i(x) \le 0 \quad i=1,\dots,m$ (convex)

} quasiconvex optimization problem

assume that p^* is the optimal value, $p^* = f_0(x^*)$.

find x
subject to $\phi_t(x) \le 0$ $\longleftrightarrow$ $f_0(x) \le t$
$f_i(x) \le 0 \quad i=1,\dots,m$

} convex feasibility problem

if the second problem has a solution then we know that $p^* \le t$. if not (i.e., the second problem is infeasible) then $p^* \ge t$.

(can be used as basis of simple algorithm for solving quasiconvex optimization problems using bisection)

$$
\begin{array}{ll}
\text{find} & x \\
\text{subject to} & \phi_t(x) \leq 0 \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b,
\end{array}
$$

Algorithm *Bisection method for quasiconvex optimization.*

given $l \leq p^\star$, $u \geq p^\star$, tolerance $\epsilon > 0$.

repeat

1. $t := (l + u)/2$.
2. Solve the convex feasibility problem (4.26).
3. **if** (4.26) is feasible, $u := t$; **else** $l := t$.

until $u - l \leq \epsilon$.

The interval $[l, u]$ is guaranteed to contain $p^\star$, *i.e.*, we have $l \leq p^\star \leq u$ at each step. In each iteration the interval is divided in two, *i.e.*, bisected, so the length of the interval after k iterations is $2^{-k}(u - l)$, where $u - l$ is the length of the initial interval. It follows that exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations are required before the algorithm terminates. Each step involves solving the convex feasibility problem on top of slide

* optimization problems

- an optimization problem in standard form is expressed as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0 \quad i=1,\dots,m \\ & h_i(x) = 0 \quad i=1,\dots,p \end{array}$$

aside: we make no assumptions on convexity at this point.

$f_0 : \mathbb{R}^n \to \mathbb{R}$ objective function

$f_i : \mathbb{R}^n \to \mathbb{R}$ inequality constraint functions

$h_i : \mathbb{R}^n \to \mathbb{R}$ equality constraint functions

$$\mathcal{D} = \left(\bigcap_{i=0}^{m} \operatorname{dom} f_i \right) \cap \left(\bigcap_{i=1}^{p} \operatorname{dom} h_i \right)$$

domain of the optimization problem

a point $x \in D$ is feasible if it satisfies the constraints

$$f_i(x) \le 0 \qquad i = 1, \cdots, m$$
$$h_i(x) = 0 \qquad i = 1, \cdots, p$$

the optimization problem is feasible if there exists at least one feasible point; it is infeasible otherwise.

the set of all feasible points is called the feasible set or the constraint set.

the optimal value of the problem is

$$p^* = \inf \left\{ f_0(x) \;\middle|\; f_i(x) \le 0 \;\; i = 1, \cdots, m, \;\; h_i(x) = 0, \; i = 1, \cdots, p \right\}$$

if the problem is infeasible, we set $p^* = +\infty$.

($x \in D$ is implied here)

- we say x^* is an optimal point, or solves the optimization problem, if x^* is feasible and $f_0(x^*) = p^*$.

  a feasible point x with $f_0(x) \le p^* + \varepsilon$ $(\varepsilon > 0)$ is called ε-suboptimal.

- we say a feasible point x is locally optimal if there is an $R > 0$ such that

$$f_0(x) = \inf \{ f_0(z) \mid f_i(z) \le 0,\ i = 1, \cdots, m,\ h_i(z) = 0,\ i = 1, \cdots, p,\ \|z - x\|_2 \le R \}$$

or in other words, x solves the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \text{subject to} & f_i(z) \le 0 \quad i = 1, \cdots, m \\ & h_i(z) = 0 \quad i = 1, \cdots, p \\ & \|z - x\|_2 \le R \end{array}$$

(this means that x minimizes f_0 over nearby points.)

- the term *globally optimal* is used to distinguish between *locally optimal* and *optimal*. in this course, *optimal* will mean *globally optimal*.

- if x is feasible and $f_i(x) = 0$ we say the ith inequalit constraint $f_i(x) \leq 0$ is *active* at x. if $f_i(x) < 0$ we say th constraint $f_i(x) \leq 0$ is *inactive*. (equality constraints are always active at all feasible points.)

- the *feasibility problem*

$$\begin{array}{lll} \text{find} & x & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

determines if the constraints are consistent, and if so, to find a point that satisfies them.

- example: consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & l_i \le x_i \le u_i \qquad i=1,\dots,n \end{array}$$

in standard form it becomes

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & l_i - x_i \le 0 \qquad i=1,\dots,n \\ & x_i - u_i \le 0 \qquad i=1,\dots,n \end{array}$$



- to solve the maximization problem

$$\begin{array}{ll} \text{maximize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0 \qquad i=1,\dots,m \\ & h_i(x) = 0 \qquad i=1,\dots,p \end{array}$$

we minimize $-f_0$ subject to the same constraints.

- equivalent problems: we call two problems equivalent if from a solution of one we can easily find a solution of the other.

- example: consider

$$\begin{aligned} \text{minimize} \quad & \tilde{f}_0(x) = \alpha_0 f_0(x) \\ \text{subject to} \quad & \tilde{f}_i(x) = \alpha_i f_i(x) \le 0 \qquad i = 1, \dots, m \\ & \tilde{h}_i(x) = \beta_i h_i(x) = 0 \qquad i = 1, \dots, p \end{aligned}$$

where $\alpha_i > 0$, $i = 0, \dots, m$, and $\beta_i \neq 0$, $i = 1, \dots, p$.

the feasible sets of this problem and the original problem are the same. and a point is optimal for the original problem iff it is optimal for the scaled one → equivalence.

- change of variables: suppose $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one, with image/range covering the problem domain D, i.e., $\varphi(\mathrm{dom}\,\varphi) \supseteq D$. define functions $\tilde{f}_i$ and $\tilde{h}_i$ as

$$\tilde{f}_i(z) = f_i(\varphi(z)), \quad i=1,\cdots,m \qquad \tilde{h}_i(z) = h_i(\varphi(z)), \quad i=1,\cdots,p$$

the problem

$$\begin{aligned} \text{minimize} \quad & \tilde{f}_0(z) \\ \text{subject to} \quad & \tilde{f}_i(z) \le 0 \quad i=1,\cdots,m \\ & \tilde{h}_i(z) = 0 \quad i=1,\cdots,p \end{aligned}$$

with variable z, is related to the original problem through the change of variable (or substitution) $x = \varphi(z)$.

the original and the transformed problems are equivalent.
if x solves the original, then $z = \varphi^{-1}(x)$ solves the transformed
if z solves the transformed, then $x = \varphi(z)$ solves the original.

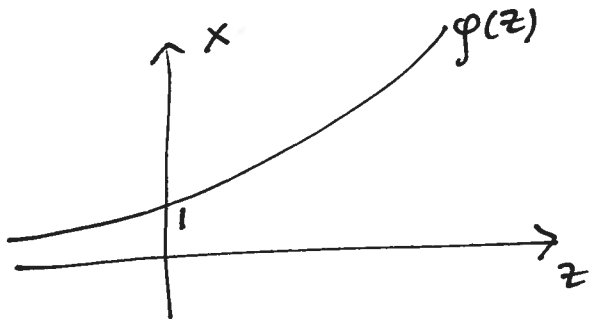
• example: consider

$$\begin{aligned} &\text{minimize} && f_0(x) \\ &\text{subject to} && f_i(x) \le 1 \quad i=1,\dots,m \\ &&& g_i(x) = 1 \quad i=1,\dots,p \end{aligned}$$

$D: x > 0$

and take $\varphi(z) = e^z$

(recall $D = \{x > 0\}$)



(one-to-one correspondence between every $x > 0$ and z.)

the original problem is equivalent to

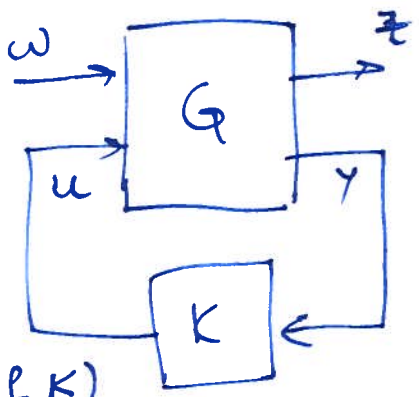
$$\begin{aligned} &\text{minimize} && f_0(e^z) \\ &\text{subject to} && f_i(e^z) \le 1 \quad i=1,\dots,m \\ &&& g_i(e^z) = 1 \quad i=1,\dots,p \end{aligned}$$

- example: an important problem in optimal control

$$\min_{K} \; \| G_{zw} + G_{zu} K (I - G_{yu} K)^{-1} G_{yw} \|_{\mathcal{H}_2}$$

$$\text{s.t.} \quad K \in \mathcal{S}$$

$\mathcal{S}$ → subspace (that encodes special structure of K)



---

scalar case

$$Q = K(1 - G_{yu} K)^{-1} \quad \longleftrightarrow \quad K = \underbrace{Q(1 + G_{yu} Q)^{-1}}_{\varphi(Q)}$$

φ is one-to-one

---

problem becomes

$$\min_{Q} \; \| G_{zw} + G_{zu} Q G_{yu} \|_{\mathcal{H}_2}$$

$$\text{s.t.} \quad \varphi(Q) \in \mathcal{S}$$

} equivalent to original optz problem.

objective $\sim$ $h(g(Q))$ convex

h: convex, g: affine

"quadratic invariance"

gives a condition on subspace $\mathcal{S}$ such that

$$\varphi(Q) \in \mathcal{S} \iff Q \in \mathcal{S}$$

(namely, that $Q G_{yu} Q \in \mathcal{S}$ if & only if $Q \in \mathcal{S}$)

↓ for such constraint sets $\mathcal{S}$, the optimization problem now becomes

$$\left.\begin{array}{ll} \min\limits_{Q} & \| G_{zw} + G_{zu} Q G_{yw} \|_{\mathcal{H}_2} \\ \text{s.t.} & Q \in \mathcal{S} \end{array}\right\}$$ convex, find Q^*, then

$$K^* = Q^* (I + G_{yu} Q^*)^{-1}.$$

- transformation of objective and constraint functions: suppose that $\psi_0 : \mathbb{R} \to \mathbb{R}$ is monotone increasing, $\psi_1, \cdots, \psi_m : \mathbb{R} \to \mathbb{R}$ satisfy $\psi_i(u) \leq 0$ iff $u \leq 0$, and $\psi_{m+1}, \cdots, \psi_{m+p} : \mathbb{R} \to \mathbb{R}$ satisfy $\psi_i(u) = 0$ iff $u = 0$. define

$$\tilde{f}_i(x) = \psi_i(f_i(x)) \quad i = 0, \cdots, m \qquad \tilde{h}_i(x) = \psi_{m+i}(h_i(x)) \quad i = 1, \cdots, p$$

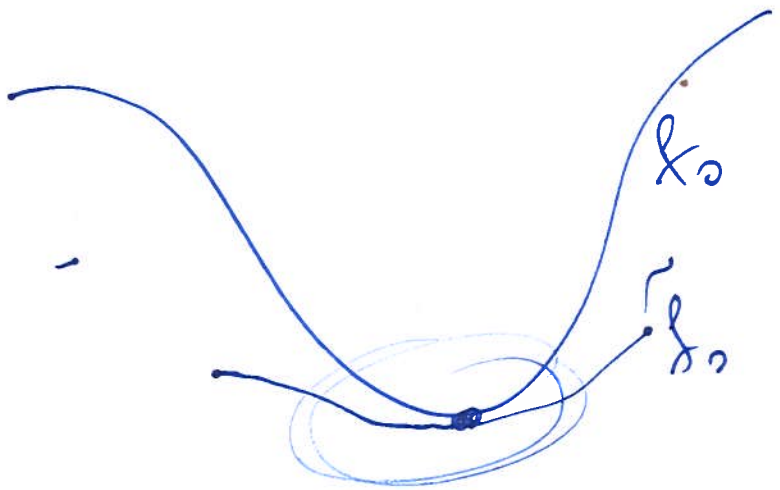
then the problem

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x) \\ \text{subject to} & \tilde{f}_i(x) \leq 0 \quad i = 1, \cdots, m \\ & \tilde{h}_i(x) = 0 \quad i = 1, \cdots, p \end{array}$$

and the original problem are equivalent; indeed the feasible set & the optimal points are identical.

aside:

$$\tilde{f}_0(x) = \tilde{\psi}_0(f_0(x))$$



$$\frac{\partial \psi_0(f_0(x))}{\partial x} = \underbrace{\frac{\partial \psi}{\partial f_0}}_{>0} \frac{\partial f_0}{\partial x}$$

monotonicity guarantees that slopes are preserved

- example: consider the optimization problem

$$\text{minimize} \quad \|Ax-b\|_2$$

with $x \in \mathbb{R}^n$. we can instead solve the problem

$$\text{minimize} \quad \|Ax-b\|_2^2 = (Ax-b)^T(Ax-b)$$

which corresponds to $\psi_0(u) = u^2$.

these problems are equivalent and the optimal points are the same.

however the two problems are not the same; in particular, the objective in the original problem is not differentiable at any point x for which $Ax-b=0$, whereas the objective in the transformed problem is quadratic & differentiable everywhere.

- example: recall $\max. \; x_1 x_2$ s.t. $f(x) \leq 0$, $x \succeq 0$ which we rewrote equivalently as

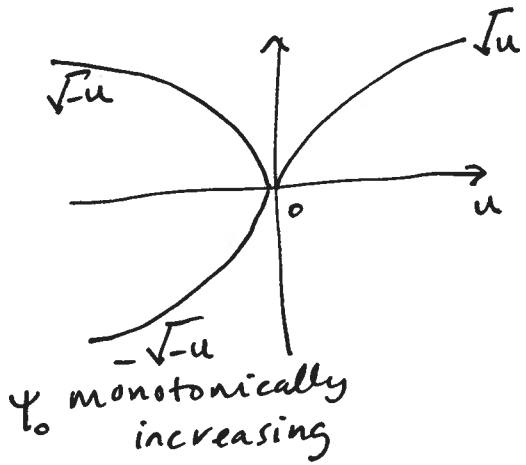
$$\max. \; +\alpha^2 \quad \text{s.t.} \quad \begin{bmatrix} x_1 & \alpha \\ \alpha & x_2 \end{bmatrix} \succeq 0, \; f(x) \leq 0, \; x \succeq 0$$

$$\updownarrow$$

$$\min. \; -\alpha^2 \quad \text{s.t.} \quad \begin{bmatrix} x_1 & \alpha \\ \alpha & x_2 \end{bmatrix} \succeq 0, \; f(x) \leq 0, \; x \succeq 0$$

$$\updownarrow \quad \psi_0(u) = -\sqrt{-u}, \; f_0(\alpha, x) = -\alpha^2$$

$$\min. \; -|\alpha| \quad \begin{bmatrix} x_1 & \alpha \\ \alpha & x_2 \end{bmatrix} \succeq 0, \; f(x) \leq 0, \; x \succeq 0$$



ψ_0 monotonically increasing

$$\longleftrightarrow \quad \max. \; \alpha \quad \text{s.t.} \quad \begin{bmatrix} x_1 & \alpha \\ \alpha & x_2 \end{bmatrix} \succeq 0, \; f(x) \leq 0, \; x \succeq 0, \; \alpha \geq 0$$

- example: consider minimize $f_0(x)$

  subject to $f_i(x) \le 1 \quad i=1,\dots,m$

  $g_i(x) = 1 \quad i=1,\dots,p$

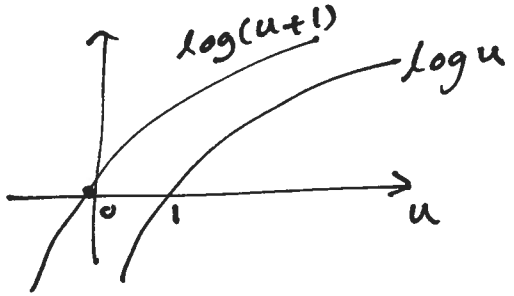
$\psi_0(u) = \log(u)$

ψ_0 is monotonic increasing

$f_i(x) \le 1 \leftrightarrow f_i(x) - 1 \le 0 \leftrightarrow \log(f_i(x) - 1 + 1) \le 0 \leftrightarrow \log(f_i(x)) \le 0$

$\psi_i(u) = \log(u+1)$

$\begin{cases} \psi_i(u) \le 0 \Leftrightarrow u \le 0 \\ \psi_i(u) = 0 \Leftrightarrow u = 0 \end{cases}$



$g_i(x) = 1 \leftrightarrow g_i(x) - 1 = 0 \leftrightarrow \log(g_i(x) - 1 + 1) = 0 \leftrightarrow \log(g_i(x)) = 0$

---

the original problem is equivalent to

minimize $\log(f_0(x))$

subject to $\log(f_i(x)) \le 0 \quad i=1,\dots,m$

$\log(g_i(x)) = 0 \quad i=1,\dots,p$

• example: consider

minimize $\sqrt{\frac{x}{y}}$ (not convex)

subject to $y \le \frac{1}{x}, x \ge \frac{1}{e}, y \ge \frac{1}{e}$

$\updownarrow$ $x = e^v$, $y = e^w$

minimize $\sqrt{\frac{e^v}{e^w}} = e^{\frac{v-w}{2}}$

subject to $e^w \le e^{-v}$, $e^v \ge \frac{1}{e}$, $e^w \ge \frac{1}{e}$

$e^{w+v} \le 1$

$e^{-v-1} \le 1$

$e^{-w-1} \le 1$

$\updownarrow$ $\psi_0 = \log(u)$, $\psi_1(u) = \log(u+1)$

minimize $\log\left(e^{\frac{v-w}{2}}\right)$

subject to $\log(e^{w+v}) \le 0$, $\log(e^{-v-1}) \le 0$, $\log(e^{-w-1}) \le 0$

$\updownarrow$

minimize $\frac{v-w}{2}$

subject to $v+w \le 0$, $\underbrace{-v-1 \le 0}_{v \ge -1}$, $\underbrace{-w-1 \le 0}_{w \ge -1}$

which is a "linear program"

(& definitely convex!)

$f_0 = \sqrt{\frac{x}{y}}$

$\downarrow$

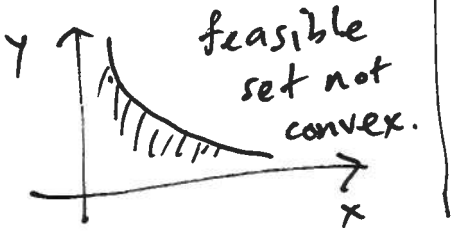
$\nabla^2 f_0 = \frac{1}{4\sqrt{xy}} \begin{bmatrix} -\frac{y}{x} & 1 \\ 1 & -\frac{x}{y} \end{bmatrix}$

$\downarrow$ $z = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow z^T \nabla^2 f_0 z < 0$

$\nabla^2 f_0 \not\succeq 0$

objective not a convex function.

$y \le \frac{1}{x}$



- slack variables: since $f_i(x) \leq 0$ iff there is an $s_i \geq 0$ that satisfies $f_i(x) + s_i = 0$, we obtain the equivalent problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & s_i \geq 0 & i = 1, \ldots, m \\ & f_i(x) + s_i = 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

the variable s_i is called the slack variable associated with the original inequality constraint $f_i(x) \leq 0$

note that we have added m new optimization variables.

– eliminating equality constraints: if we can explicitely parameterize all solutions of the equality constraint

$$h_i(x) = 0 \qquad i = 1, \dots, p$$

using some parameter $z \in \mathbb{R}^k$, then we can eliminate all equality constraints. suppose $\varphi : \mathbb{R}^k \to \mathbb{R}^n$ is such that x satisfies $h_i(x) = 0 \ \forall i$ iff there is some $z \in \mathbb{R}^k$ such that $x = \varphi(z)$.

then

$$\begin{aligned} \text{minimize} \quad & \tilde{f}_0(z) = f_0(\varphi(z)) \\ \text{subject to} \quad & \tilde{f}_i(z) = f_i(\varphi(z)) \leq 0 \qquad i = 1, \dots, m \end{aligned}$$

is equivalent to the original problem.

if z is optimal for the transformed problem then $x = \varphi(z)$ is optimal for the original problem, and vice versa.

- example: consider $h: \mathbb{R}^2 \to \mathbb{R}$ defined as

$$h(x) = x_1 x_2 - 1$$

then $\varphi: \mathbb{R} \to \mathbb{R}^2$ given by

$$\varphi(z) = \begin{bmatrix} z \\ 1/z \end{bmatrix}$$

satisfies

$$h(\varphi(z)) = 0 \qquad \forall z \neq 0$$

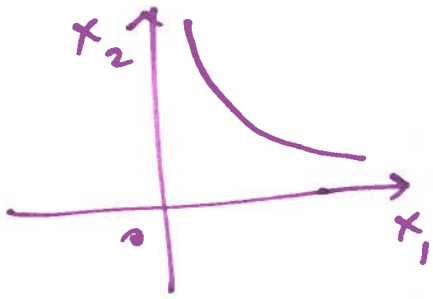
$h(x) = 0$

$\downarrow$

$x_1 x_2 - 1 = 0$

$\downarrow$

$x_2 = \frac{1}{x_1}$



- eliminating linear equality constraints: suppose all equality constraints are linear, i.e., we have $Ax = b$. if $Ax = b$ is inconsitent, i.e., $b \notin \mathcal{C}(A)$, then the optimization problem in infeasible.

assuming this is not the case and $b \in \mathcal{C}(A)$, let x_0 denote a solution. let $F \in \mathbb{R}^{n \times k}$ be any matrix with $\mathcal{C}(F) = \mathcal{N}(A)$. then the general solution of $Ax = b$ is given by $Fz + x_0$, where $z \in \mathbb{R}^k$. the problem

$$\begin{aligned} &\text{minimize} && f_0(Fz + x_0) \\ &\text{subject to} && f_i(Fz + x_0) \leq 0 \quad i = 1, \ldots, m \end{aligned}$$

is equivalent to the original problem, and has rank (A) fewer optimization variables.

$f_1 \cdots f_k$

$$x = z_1 f_1 + \cdots + z_k f_k + x_0$$

$$= \begin{pmatrix} | & & | \\ f_1 & \cdots & f_k \\ | & & | \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} + x_0$$

$$= Fz + x_0$$

$$\mathcal{C}(F) = N(A).$$

- introducing equality constraints: we can also introduce equality constraints and new variables into a problem. for example

$$\begin{aligned} \text{minimize} \quad & f_0(A_0x+b_0) \\ \text{subject to} \quad & f_i(A_ix+b_i) \leq 0 \qquad i=1,\cdots,m \\ & h_i(x)=0 \qquad i=1,\cdots,p \end{aligned}$$

is equivalent to

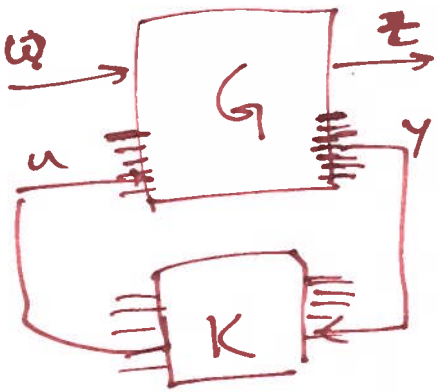
$$\begin{aligned} \text{minimize} \quad & f_0(y_0) \\ \text{subject to} \quad & f_i(y_i) \leq 0 \qquad i=1,\cdots,m \\ & y_i = A_ix+b_i \qquad i=0,\cdots,m \\ & h_i(x)=0 \qquad i=1,\cdots,p \end{aligned}$$

• example: recall the optimal control problem

$$\underset{K}{\text{minimize}} \quad \| G_{zw} + G_{zu} K (I - G_{yu} K)^{-1} G_{yw} \|_{\mathcal{H}_2} + \gamma \, \text{card}(K)$$

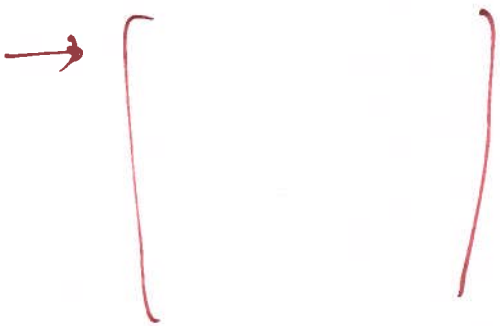
$$\underset{K,Q}{\text{minimize}} \quad \| G_{zw} + G_{zu} K (I - G_{yu} K)^{-1} G_{yw} \|_{\mathcal{H}_2} + \gamma \, \text{card}(Q)$$

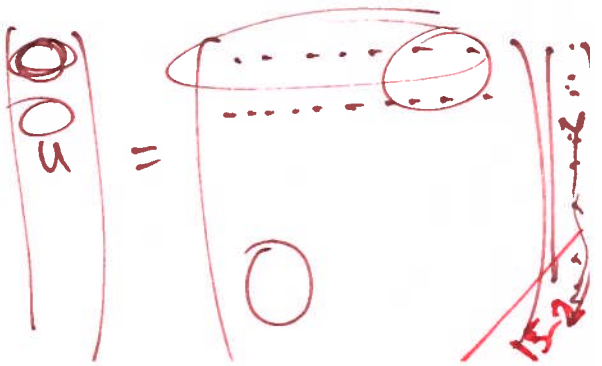
subject to $K = Q$



$\| \cdot \|_{\mathcal{H}_2} \sim f(K)$ $\qquad \gamma \, \text{card}(\cdot) \sim g(Q)$

$$\mathcal{L} = f(K) + g(Q) + \underbrace{q_\lambda(K-Q)}_{\text{quadratic function of } K-Q}$$





now perform iteration

ADMM

minimize $\mathcal{L}$ $\underset{K}{\text{minimize}}$ $\rightarrow$ $\min_K f(K) + q_x(K-Q)$ $\rightarrow$ new K.

minimize $\mathcal{L}$ $\underset{Q}{\text{minimize}}$ $\rightarrow$ $\min_Q g(Q) + q_x(K-Q)$ $\rightarrow$ new Q

$$\min_Q \gamma \text{Card}(\Theta) + \frac{\rho}{2}\|\Theta - K^{(\lambda)}\|_F^2$$

sol: $$Q^*_{ij} = \begin{cases} 0 & |K^{(\lambda)}_{ij}| < \beta \\ K^{(\lambda)}_{ij} & |K^{(\lambda)}_{ij}| \geq \beta \end{cases}$$

$$\beta = \sqrt{\frac{2\gamma}{\rho}}$$

update coefficients λ in $q_x(\cdot)$ based on the new K & Q.

15-2

- optimizing over some variables: we always have

$$\inf_{x,y} f(x,y) = \inf_{x} \tilde{f}(x)$$

where

$$\tilde{f}(x) = \inf_{y} f(x,y)$$

in other words, we can always minimize a function by first minimizing over some variables and then minimizing over the rest.

- example: consider

$$\begin{aligned} &\text{minimize} && x^T A x + 2x^T B y + y^T C y \\ &\text{subject to} && f_i(x) \le 0 \qquad i = 1, \dots, m \end{aligned}$$

where A, C are symmetric and $C \succ 0$. we can (analytically) minimize over y to obtain

$$\inf_y \left(x^T A x + 2x^T B y + y^T C y \right) = x^T (A - B C^{-1} B^T) x .$$

therefore the original problem is equivalent to

$$\begin{aligned} &\text{minimize} && x^T (A - B C^{-1} B^T) x \\ &\text{subject to} && f_i(x) \le 0 \qquad i = 1, \dots, m . \end{aligned}$$

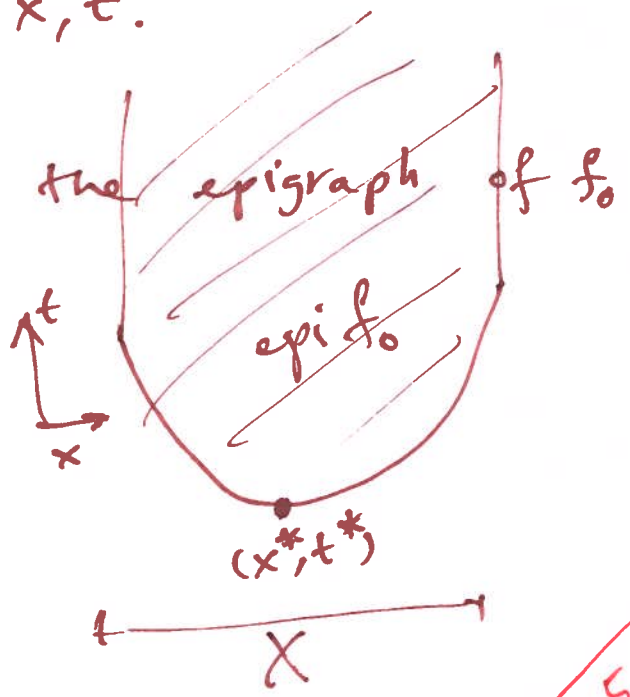
- epigraph form: the epigraph form of the standard problem is

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x) - t \le 0 \\ & f_i(x) \le 0 \qquad i = 1, \dots, m \\ & h_i(x) = 0 \qquad i = 1, \dots, p \end{array}$$

with variables $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. → ((x,t) is optimal here iff x is optimal for standard problem & $t = f_0(x)$)

note that the objective of the epigraph form is a linear function of x, t.

geometrically, we minimize t over the epigraph of f_0 subject to the constraints on x



aside: the epigraph form shows that it is almost meaningless to say that an opt'z problem has a convex objective.

• recall example

$$\begin{aligned} &\text{maximize} && x_1 x_2 \\ &\text{subject to} && f(x) \leq 0 \\ & && x \geq 0 \end{aligned}$$

in epigraph form we get, equivalently

$$\begin{aligned} &\text{maximize} && t \qquad (*) \\ &\text{subject to} && t - x_1 x_2 \leq 0 \\ & && f(x) \leq 0 \\ & && x \geq 0 \end{aligned}$$

(equivalently

$$\begin{aligned} &\text{min.} && -x_1 x_2 \\ &\text{s.t.} && \dots \end{aligned} \quad \leftrightarrow \quad \begin{aligned} &\text{min.} && s \\ &\text{s.t.} && -x_1 x_2 - s \leq 0 \\ & && \dots \end{aligned} \quad \leftrightarrow \quad \begin{aligned} &\text{max.} && -s \\ &\text{s.t.} && -s - x_1 x_2 \leq 0 \\ & && \dots \end{aligned}$$

$\updownarrow$ $t = -s$

$$\begin{aligned} &\text{max.} && t \\ &\text{s.t.} && t - x_1 x_2 \leq 0 \\ & && \dots \end{aligned}$$

which is the same as (*).)

- constraint specification: we can include the constraints implicitly in the objective function thru its domain. the standard form can be expressed as the unconstrained problem

$$\text{minimize} \quad F(x)$$

$$\text{dom } F = \{ x \in \text{dom } f_0 \mid f_i(x) \leq 0,\ i=1,\dots,m,\ h_i(x)=0,\ i=1,\dots,p \}$$

and $F(x) = f_0(x)$ for $x \in \text{dom } F$.

equivalently, we can define F to have value $+\infty$ for x not feasible. we can also make this explicit by using the indicator function, as shown in the next example.

- example: we will see how this naturally arises when we consider the "dual" of a "linear program"

- example: for a set $C \subseteq \mathbb{R}^n$, we can define the indicator function $I_C(x)$ by

$$I_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases}$$

then, the standard problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \in C \end{array}$$

is equivalent to the unconstrained problem

$$\text{minimize} \quad f_0(x) + I_C(x).$$



aside: this becomes extremely powerful in conjunction with ADMM

$$\min. \; f_0(x) \quad \text{s.t.} \quad x \in C$$

$$\longleftrightarrow \quad \min. \; f_0(x) + I_C(x) \quad \longleftrightarrow \quad \min. \; f_0(x) + I_C(y) \quad \text{s.t.} \quad x = y$$

$$\mathcal{L} = f_0(x) + I_C(y) + q_\lambda(x - y)$$

(can be interpreted as finding min. dist. between x & set C)

now perform iteration

ADMM:

- minimize $\mathcal{L}$ $\longrightarrow$ $\min_x \; f_0(x) + q_\lambda(x - y)$
- minimize $\mathcal{L}$ $\longrightarrow$ $\min_y \; I_C(y) + q_\lambda(x - y)$ $\longleftrightarrow$ $\min_y \; q_\lambda(x - y)$ s.t. $y \in C$
- update λ based on x & y.

aside: we will see later...

Inequality constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{1}$$

- f_i convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- we assume $p^\star$ is finite and attained
- we assume problem is strictly feasible: there exists $\tilde{x}$ with

$$\tilde{x} \in \mathbf{dom}\, f_0, \qquad f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \qquad A\tilde{x} = b$$

hence, strong duality holds and dual optimum is attained

aside: we will see later...

Logarithmic barrier

reformulation of (1) via indicator function:

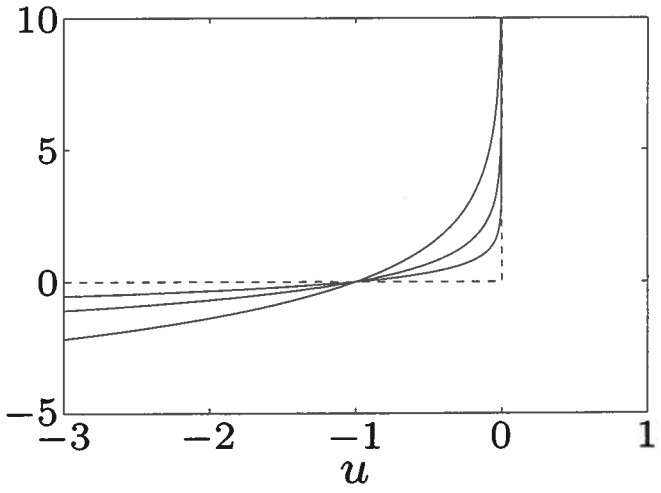
$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

where $I_-(u) = 0$ if $u \leq 0$, $I_-(u) = \infty$ otherwise (indicator function of $\mathbf{R}_-$)

approximation via logarithmic barrier

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t) \sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of I_-
- approximation improves as $t \to \infty$



- upper and lower bounds: consider the two problems

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X \end{array} \qquad \text{and} \qquad \begin{array}{ll} \text{minimize} & g(y) \\ \text{subject to} & y \in Y \end{array}$$

with optimal values p^* and q^*, respectively.

suppose there exists $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ such that for $x \in X$, $y = \varphi(x)$ satisfies $y \in Y$ with $g(y) = f(x)$ [which implies $p^* \geq q^*$],

and there exists $\psi : \mathbb{R}^m \to \mathbb{R}^n$ such that for $y \in Y$, $x = \psi(y)$ satisfies $x \in X$ with $f(x) = g(y)$ [which implies $p^* \leq q^*$].

then the two optimization problems are equivalent.

- example: consider (the linear-fractional program)

$$\begin{aligned} \text{minimize} \quad & f_0(x) = \frac{c^T x + d}{e^T x + f} \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned} \qquad (1)$$

$$\text{dom } f_0 = \{ x \mid e^T x + f > 0 \}.$$

the objective is quasiconvex (and quasiconcave, hence quasilinear). this problem can be transformed to an equivalent (linear program)

$$\begin{aligned} \text{minimize} \quad & c^T y + dz \\ \text{subject to} \quad & Gy - hz \preceq 0 \\ & Ay - bz = 0 \\ & e^T y + fz = 1 \\ & z \geq 0 \end{aligned} \qquad (2)$$

if x is feasible in ① then

$$y = \frac{x}{e^T x + f}, \qquad z = \frac{1}{e^T x + f}$$

is feasible in ②

$$\left(\begin{array}{l} Gx \preceq h, \quad e^T x + f > 0 \\ G \dfrac{x}{e^T x + f} \preceq h \dfrac{1}{e^T x + f} \rightarrow Gy \preceq hz \end{array} \right)$$

and with the same objective value $c^T y + dz = f_0(x)$

$$\left(\frac{c^T x + d}{e^T x + f} = \frac{c^T x}{e^T x + f} + \frac{d}{e^T x + f} = c^T y + dz \right)$$

it follows that the optimal value of ① is greater than or equal to the optimal value of ②.

conversely, if (y,z) is feasible in ②, with $z \neq 0$, then $x = y/z$ is feasible in ①, with the same objective value $f_0(x) = c^T y + dz$

$$\left(c^T y + dz = c^T x z + dz = (c^T x + d) z \right.$$

on the other hand

$$e^T y + fz = 1 \rightarrow (e^T x + f) z = 1 \rightarrow z = \frac{1}{e^T x + f}$$

together, we get

$$\left. c^T y + dz = \frac{c^T x + d}{e^T x + f} \right)$$

we conclude that the optimal value of ① is less than or equal to the optimal value of ②.

(the case for $z = 0$ is more complex & can be found in the book.)

- convex optimization problems (in standard form): a convex optimization problem is one of the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0 \quad i=1,\dots,m \\ & a_i^T x = b_i \quad i=1,\dots,p \end{array}$$

where $f_0,\dots,f_m$ are convex functions (with convex domains).

the feasible set of a convex optimization problem is convex and is given by

$$\left(\bigcap_{i=0}^{m} \operatorname{dom} f_i\right) \cap \left(\bigcap_{i=1}^{m} \{x \mid f_i(x) \le 0\}\right) \cap \left(\bigcap_{i=1}^{p} \{x \mid a_i^T x = b_i\}\right)$$

- in a convex optimization problem we minimize a convex function over a convex set.

- we will also refer to

$$\begin{aligned} &\text{maximize} && f_0(x) \\ &\text{subject to} && f_i(x) \le 0 \quad i=1,\dots,m \\ &&& a_i^T x = b_i \quad i=1,\dots,p \end{aligned}$$

as a convex optimization problem if the objective function is concave and the inequality constraint functions are convex.

$$\updownarrow$$

$$\begin{aligned} &\text{minimize} && -f_0(x) \\ &\text{subject to} && \vdots \quad \} \text{ same as above} \end{aligned}$$

- abstract form convex optimization: consider

$$\begin{aligned} \text{minimize} \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} \quad & f_1(x) = \frac{x_1}{1+x_2^2} \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{aligned}$$

which is not a convex optimization problem in standard form, since h_1 is not affine and f_1 is not convex. still, the feasible set is convex. we can reformulate the problem to the equivalent one

$$\begin{aligned} \text{minimize} \quad & f_0(x) = x_1^2 + x_2^2 \\ & \tilde{f}_1(x) = x_1 \leq 0 \\ & \tilde{h}_1(x) = x_1 + x_2 = 0 \end{aligned}$$

which is in standard convex optimization form.

- in the optimization literature, the term "abstract convex optimization problem" is sometimes used to describe the problem of minimizing a convex function over a convex set.

  in this course, a convex optimization problem requires more: it is also required that inequality functions are convex and the equality functions are affine.

- a fundamental property of convex optimization problems is that any locally optimal point is also (globally) optimal.

- an optimality criterion for differentiable f_0: suppose that f_0 in a convex optimization problem is differentiable, so that for all x, y

$$f_0(y) \geq f_0(x) + (\nabla f_0(x))^T (y - x). \qquad (1)$$

Let X denote the feasible set,

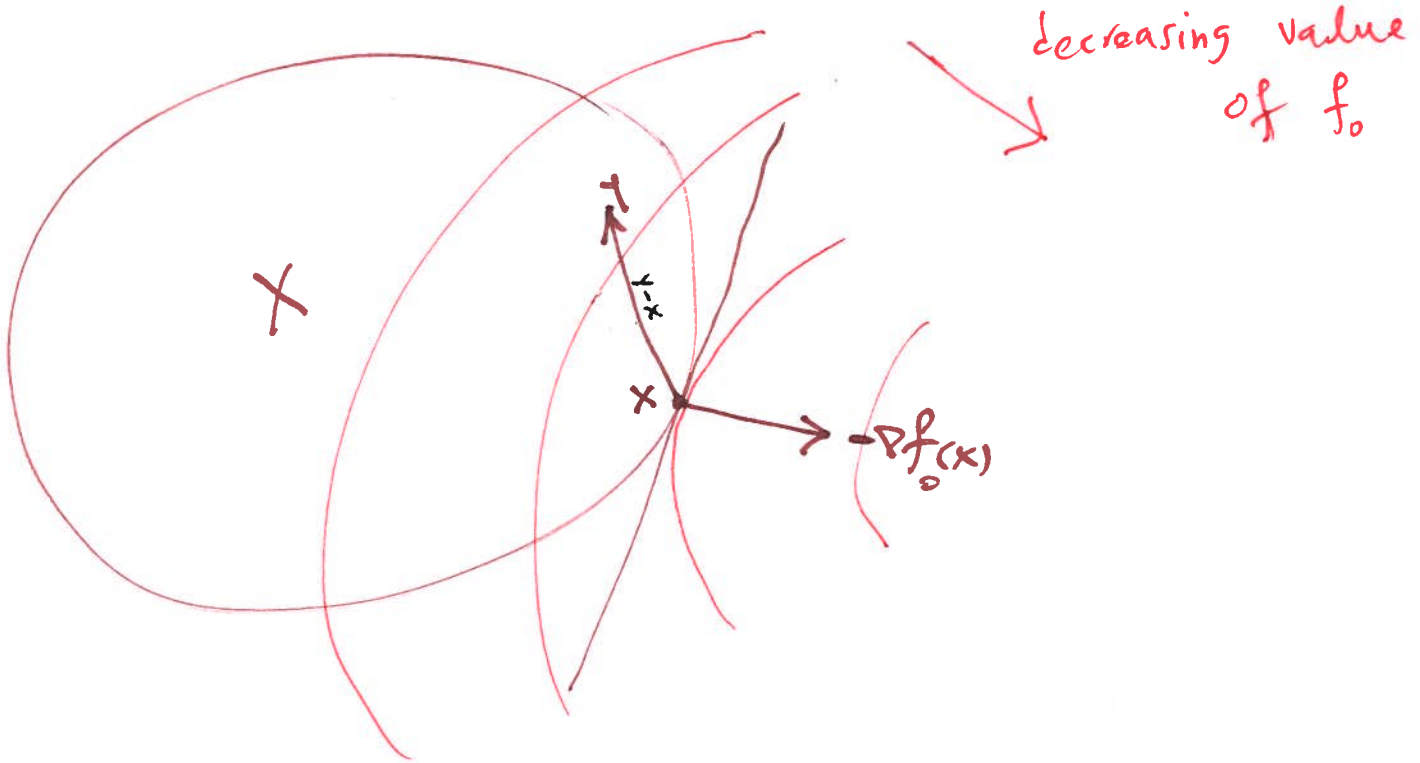
$$X = \{ x \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p \}.$$

then x is optimal if and only if $x \in X$ and

$$(\nabla f_0(x))^T (y - x) \geq 0 \text{ for all } y \in X. \qquad (2)$$

(geometrically, if $\nabla f_0(x) \neq 0$, this means that $-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at x.)

(note that this is not a particularly useful optimality condition from a computational point of view)



proof:

($\Rightarrow$) suppose $x \in X$ and satisfies $(\nabla f_0(x))^T (y-x) \geq 0 \quad \forall y \in X$.

then if $y \in X$ we have from ① that $f_0(y) \geq f_0(x)$.

this shows that x is optimal

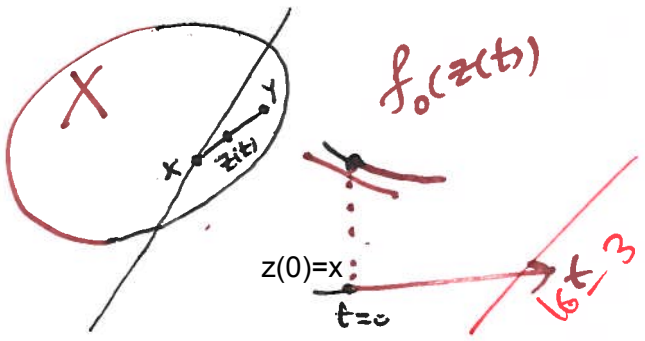
($\Leftarrow$) suppose x is optimal, but ② does not hold, i.e., for some $y \in X$ we have

$$(\nabla f_0(x))^T (y-x) < 0 .$$

consider $z(t) = ty + (1-t)x$, where $0 \le t \le 1$. because $z(t)$ is on the line segment between x & y, and X is convex, then $z(t)$ is feasible. since

$$\frac{d}{dt} f_0(z(t)) \Big|_{t=0} = (\nabla f_0(x))^T (y-x) < 0,$$

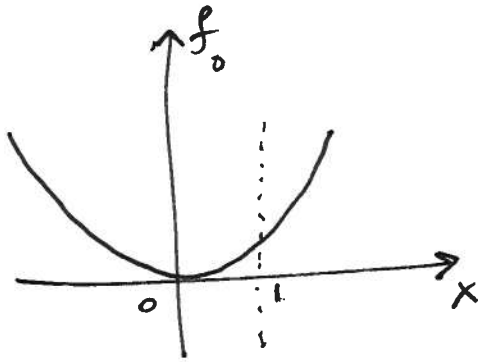
so for small positive t, we have $f_0(z(t)) < f_0(x)$, and x is not optimal.



- example: apply the optimality condition to the problem

minimize x^2

subject to $x \geq 1$



condition for optimality of x: $(\nabla f_0(x))^T (y-x) \geq 0 \quad \forall y \geq 1$

$f_0(x) = x^2 \rightarrow \nabla f_0 = 2x \rightarrow 2x(y-x) \geq 0 \quad \forall y \geq 1$

$\downarrow$

$2xy \geq 2x^2$

$\downarrow x \geq 1 > 0$

$y \geq x$

the optimality condition is therefore that $y \geq x$ for all $y \geq 1$

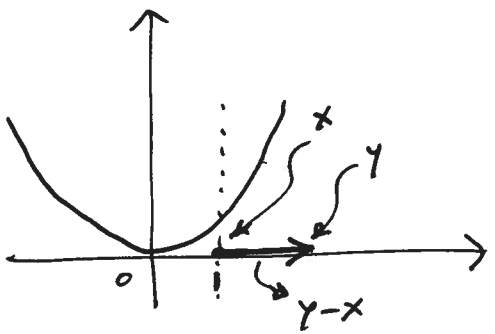
$\Downarrow$

$x = 1$ is optimal

check:

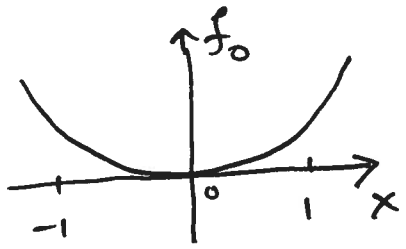
$\nabla f_0(1) = 2$

$2(y-1) \geq 0 \quad \forall y \geq 1$



- example: apply the optimality condition to the problem

minimize x^2
subject to $-1 \le x \le 1$



condition for optimality of x

$$(\nabla f_0(x))^T (y-x) \ge 0 \quad \forall\, y \in [-1,1]$$

$$\downarrow$$

$$2x(y-x) \ge 0 \quad \forall\, y \in [-1,1]$$

method ①:

if $0 \le x \le 1$ take $y=-1$ $\rightarrow$ $2x(-1-x) \ge 0$ $\rightarrow$ $x=0$
($2x \ge 0$, $(-1-x) \le 0$)

if $-1 \le x \le 0$ take $y=+1$ $\rightarrow$ $2x(+1-x) \ge 0$ $\rightarrow$ $x=0$
($2x \le 0$, $(+1-x) \ge 0$)

method ②: $y = x - t\nabla f_0(x)$, $t>0$ $\rightarrow$ $y-x = -2tx$

$\rightarrow (\nabla f_0(x))^T (y-x) = 2x(-2tx)$

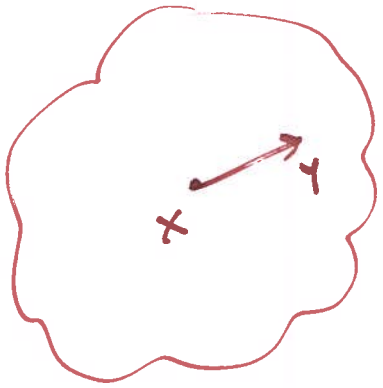
$= -4tx^2 \ge 0$ ← opt. cond.

$\downarrow t>0$

$x=0$

(note that y does not go out of $[-1,1]$; if $x=1$ then $y = 1-t \cdot 2 < 1$ and if $x=-1$ then $y = -1-t(-2) > -1$, when $t>0$)

- unconstrained problems: for an unconstrained problem, the condition $(\nabla f_0(x))^T (y-x) \geq 0 \quad \forall y \in X$, reduces to the necessary & sufficient condition



$$\nabla f_0(x) = 0$$

for x to be optimal.

to see this, suppose x is optimal, and for all feasible y we have $(\nabla f_0(x))^T (y-x) \geq 0$.

take $y = x - t\nabla f_0(x)$ $\left(\rightarrow y - x = -t\nabla f_0(x)\right)$ & $t > 0$

$$0 \leq (\nabla f_0(x))^T (y-x) = -t \|\nabla f_0(x)\|_2^2$$

which implies $\nabla f_0(x) = 0$.

(note that this is also true for a problem in which none of the constraints are active at optimal x)

- the condition $\nabla f_0(x) = 0$ can lead to several possible scenarios, based on the number of solutions

- example: consider minimizing

$$f_0(x) = \frac{1}{2} x^T P x + q^T x + r$$

where $P \succeq 0$ (which makes f_0 convex).

the necessary & sufficient condition for x to be a minimizer of f_0 is

$$\nabla f_0(x) = Px + q = 0 \longrightarrow Px = -q$$

several cases can occur.

① if $P \succ 0$, then $x^* = -P^{-1} q$

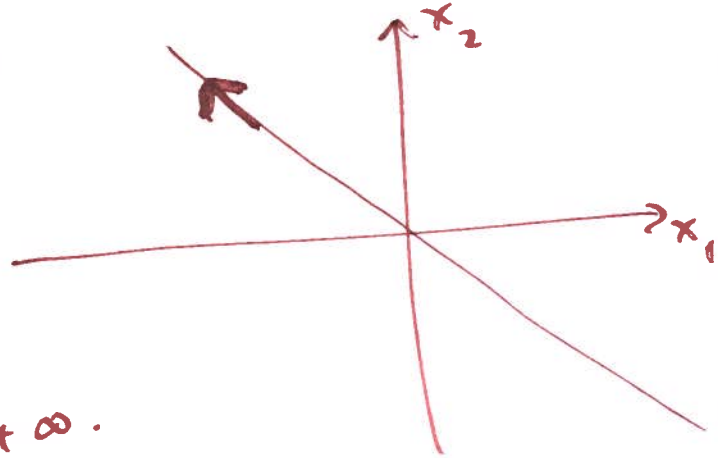
(f_0 is strictly convex here)

② if $q \notin \mathcal{C}(P)$, then there is no solution.

suppose $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \mathcal{C}(P) = \left\{ \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\}$, $\quad \mathcal{N}(P) = \left\{ \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\}$

$q = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow Px + q = 0$ has no sol. ($q \notin \mathcal{C}(P)$)

$$\frac{1}{2} x^T P x + q^T x = \frac{1}{2} (x_1 + x_2)^2 + x_1$$



$x_\alpha = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}$ $\quad \alpha > 0$

$f_0(x_\alpha) \rightarrow -\infty$ as $\alpha \rightarrow +\infty$.

③ if P is singular but $q \in \mathcal{C}(P)$, then the set of optimal points is the (affine) set

$$\{x \mid x = x_* + N(P)\}$$

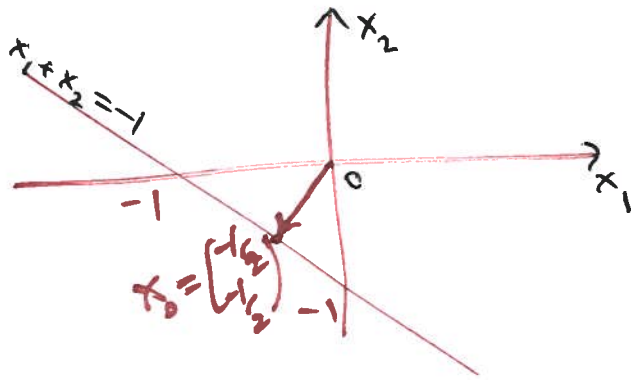
one possible choice: $-P^+ q$ $\quad P^+$: pseudoinverse of P.

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow Px + q = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \rightarrow x_1 + x_2 = -1$$

$$\frac{1}{2} x^T P x + q^T x = \frac{1}{2}(x_1 + x_2)^2 + (x_1 + x_2)$$

$$P^+ = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow -P^+ q + N(P) = -\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} = \begin{bmatrix} \alpha - 1/2 \\ -\alpha - 1/2 \end{bmatrix}$$

satisfies $x_1 + x_2 = -1$



- problems with equality constraints only: consider

$$\begin{aligned} &\text{minimize} && f_0(x) \\ &\text{subject to} && Ax = b \end{aligned} \qquad A \in \mathbb{R}^{p \times n}$$

we assume that the feasible set is not empty.

the optimality condition for a feasible x is

$$(\nabla f_0(x))^T (y - x) \geq 0 \quad \forall\, y \text{ s.t. } Ay = b.$$

since x is feasible, every y can be written as $y = x + v$, $v \in N(A)$. the optimality condition becomes

$$(\nabla f_0(x))^T v \geq 0 \quad \forall\, v \in N(A).$$

($\Rightarrow$ (a subspace if $v \in N(A)$, so is $-v$.))

which implies that $(\nabla f_0(x))^T v = 0 \quad \forall\, v \in N(A)$, or

$$\nabla f_0(x) \perp \mathcal{N}(A).$$

now, using the fact that $\mathcal{N}(A)^{\perp} = \mathcal{C}(A^T)$, this optimality condition can be expressed as

$$\nabla f_0(x) \in \mathcal{C}(A^T)$$

or, equivalently, there exists a $v \in \mathbb{R}^p$ such that

$$\nabla f_0(x) + A^T v = 0.$$

$$Ax = b$$

---

aside: $x \in \mathcal{N}(A) \rightarrow Ax = 0 \rightarrow y^T(Ax) = 0 \quad \forall y$

$\rightarrow (A^T y)^T x = 0 \quad \forall y \rightarrow \mathcal{C}(A^T) \perp \mathcal{N}(A)$

aside: we will see later...

Equality constrained minimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- f convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- we assume $p^\star$ is finite and attained

optimality conditions: $x^\star$ is optimal iff there exists a $\nu^\star$ such that

$$\nabla f(x^\star) + A^T \nu^\star = 0, \qquad Ax^\star = b$$

aside: we will see later...

equality constrained quadratic minimization (with $P \in \mathbf{S}_+^n$)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

optimality condition:

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ \nu^\star \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix} \qquad \text{(KKT/duality)}$$

- coefficient matrix is called KKT matrix
- KKT matrix is nonsingular if and only if

$$Ax = 0, \quad x \neq 0 \quad \Longrightarrow \quad x^T P x > 0$$

- equivalent condition for nonsingularity: $P + A^T A \succ 0$

- minimization over the nonnegative orthant: consider

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \succeq 0 \end{array}$$

the optimality condition for a feasible x is

$$(\nabla f_0(x))^T (y - x) \geq 0 \quad \forall\, y \succeq 0$$

the term $(\nabla f_0(x))^T y$ is unbounded below on $y \succeq 0$ unless $\nabla f_0(x) \succeq 0$.

thus we have that

$$-(\nabla f_0(x))^T x \geq -(\underbrace{\nabla f_0(x)}_{\succeq 0})^T \underbrace{y}_{\succeq 0}$$

needs to hold for all $y \succeq 0$. since the right hand side achieves its largest value for $y = 0$,

the condition reduces to

$$\left(-\nabla f_0(x)\right)^T x \geq 0.$$

but $x \geq 0$ and $\nabla f_0(x) \geq 0$, so we must have

$$\left(\nabla f_0(x)\right)^T x = 0 \quad \rightarrow \quad \sum_{i=1}^{n} \underbrace{\left(\nabla f_0(x)\right)_i}_{\geq 0} \underbrace{x_i}_{\geq 0} = 0$$

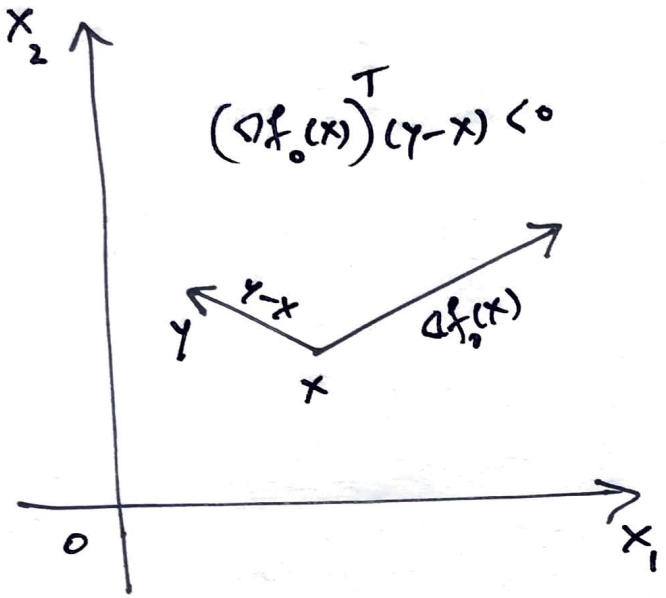
$$\downarrow$$

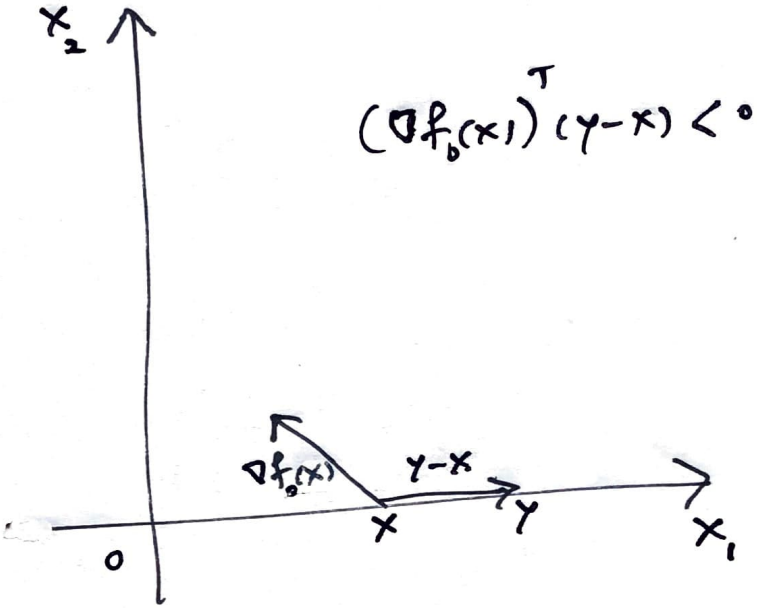
$$\left(\nabla f_0(x)\right)_i x_i = 0$$

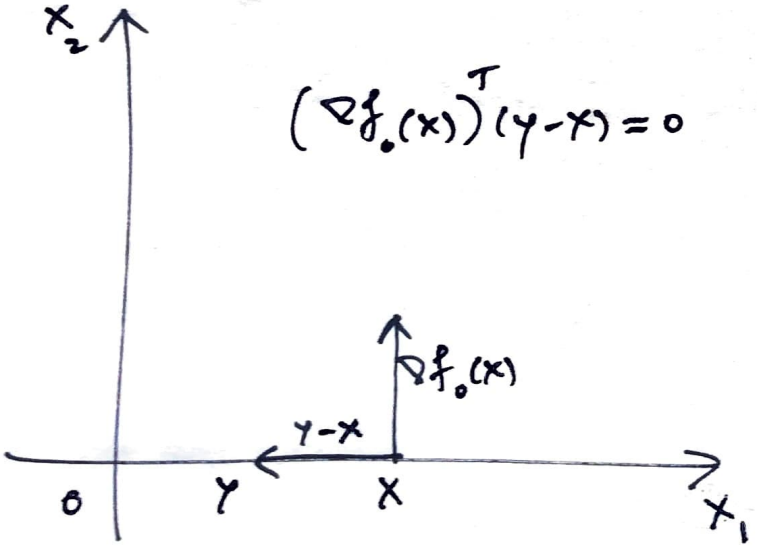
for $i = 1, \ldots, n$

the optimality conditions can thus be expressed as

$$x \geq 0 \qquad \nabla f_0(x) \geq 0 \qquad x_i \left(\nabla f_0(x)\right)_i = 0, \quad i = 1, \ldots, n$$







$$\begin{cases} \nabla f_0(x) \sim \begin{bmatrix} 0 \\ \beta \end{bmatrix} \\ x \sim \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \end{cases} \rightarrow x_i (\nabla f_0)_i = 0$$

• equivalent convex problems:

- eliminating equality constraints: for a convex problem, equality constraints are of the form $Ax = b$. we can use $x = Fz + x_0$ (where x_0 is a particular solution, & F is a matrix whose column space is the null space of A)

$$\begin{aligned} &\text{minimize} && f_0(Fz + x_0) \\ &\text{subject to} && f_i(Fz + x_0) \leq 0 \qquad i = 1, \ldots, m \end{aligned}$$

since the composition of a convex function with an affine function is convex, eliminating equality constraints preserves convexity.

- introducing equality constraints: we preserve convexity by introducing new variables and equality constraints provided the equality constraints are linear.

- slack variables: by introducing slack variables, we get $f_i(x) + s_i = 0$. we must have f_i affine to preserve convexity.

- epigraph form: the problem

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && f_0(x) \le t \quad \rightarrow \quad f_0(x) - t \le 0 \\ & && f_i(x) \le 0 \qquad i = 1, \cdots, m \\ & && a_i^T x = b_i \qquad i = 1, \cdots, p \end{aligned}$$

is convex because the objective is linear and $f_0(x) - t$ is convex in (x, t).

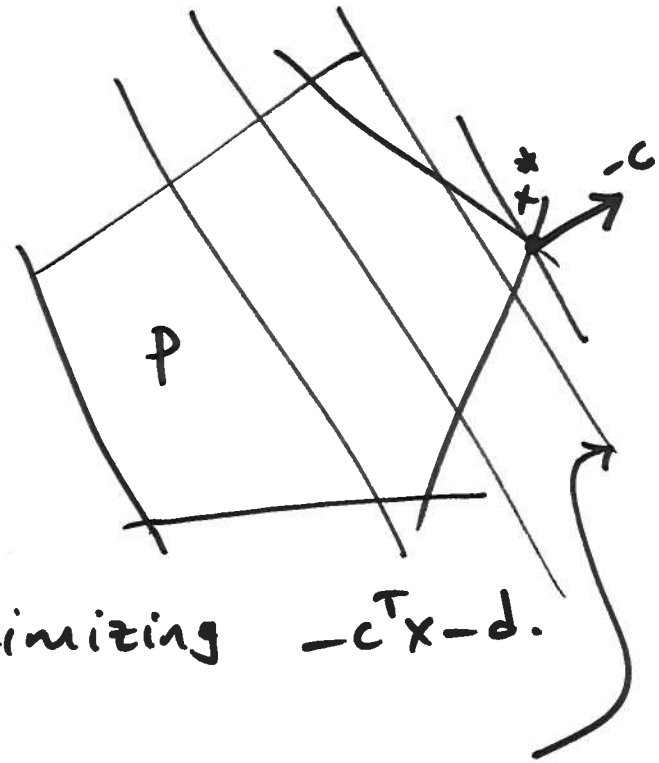
- minimizing over some variables: this preserves convexity.

(since f0(x,y) is convex in (x,y) and the feasible set is convex then minimization over y preserves convexity)

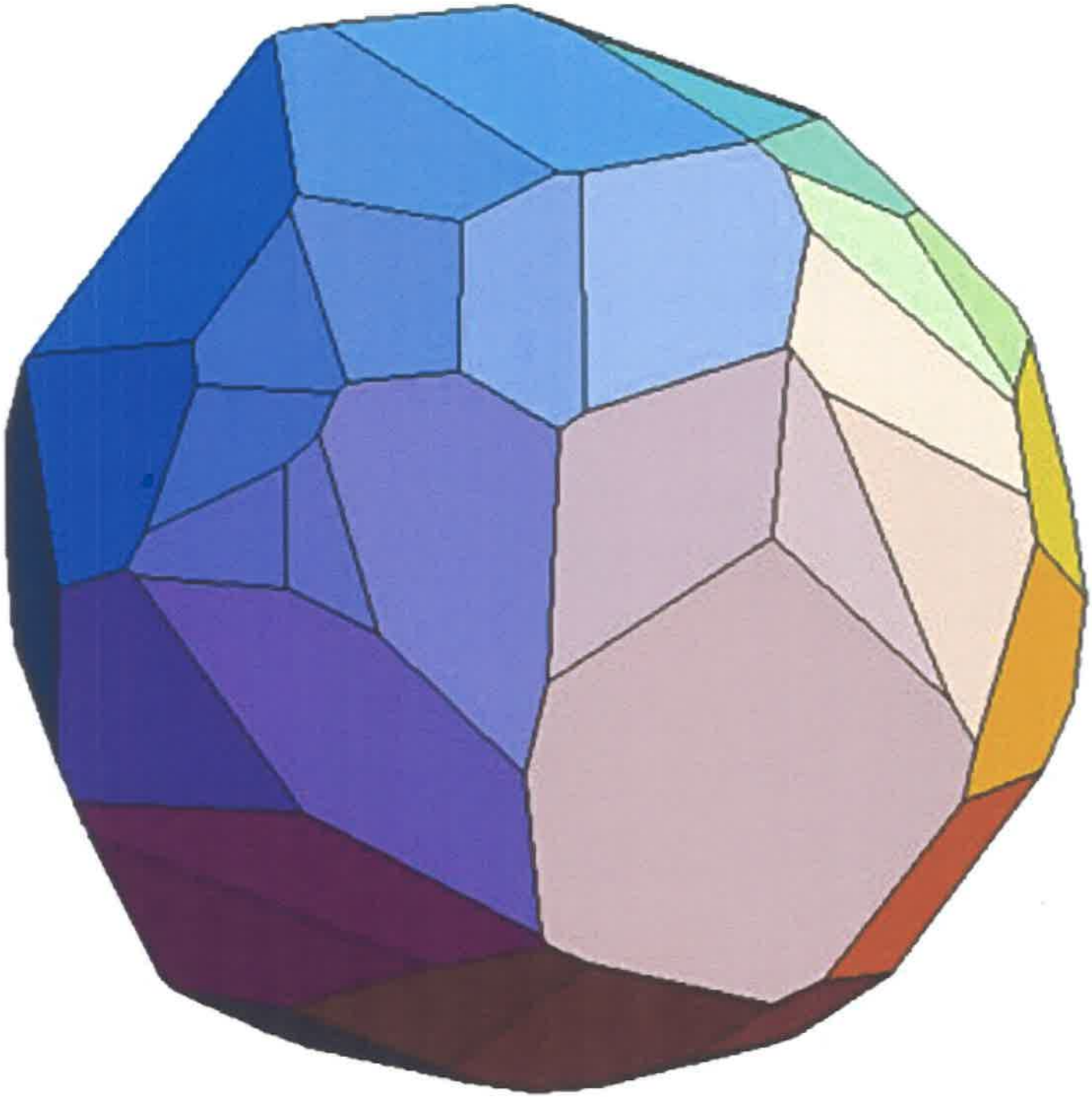
- linear programming: when the objective and constraint functions are all affine, the problem is called a linear program

$$\begin{aligned} \text{minimize} \quad & c^T x + d \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

where $G \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{p \times n}$.



- we can maximize $c^T x + d$ by minimizing $-c^T x - d$.



Linear programming

- motivating example: the "dairy" problem

① 22 gallons of milk per week

② 2 gallons of milk → 1 kilogram of butter

③ 3 gallons of milk → 1 gallon of ice cream

④ refrigerator space (to store butter) is unlimited

⑤ freezer space (to store ice cream) is 6 gallons

⑥ 1 hour of work → 4 gallons of ice cream

⑦ 1 hour of work → 1 kilogram of butter

⑧ total hours of work per week is 6

⑨ profit for ice cream is $5 per gallon

⑩ profit for butter is $4 per kilogram

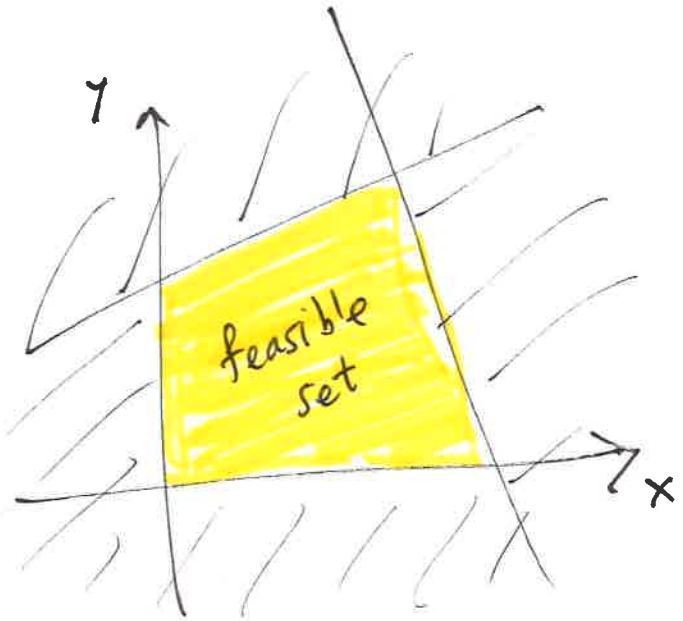
how much butter & ice cream should we make to maximize profit?

---

x: gallons of ice cream

y: kilograms of butter

$J(x,y)$: total profit.



$$\underset{x,y}{\text{maximize}} \quad J(x,y) := 5x + 4y \qquad \leftarrow ⑨ \& ⑩ \quad \text{(profit)}$$

subject to

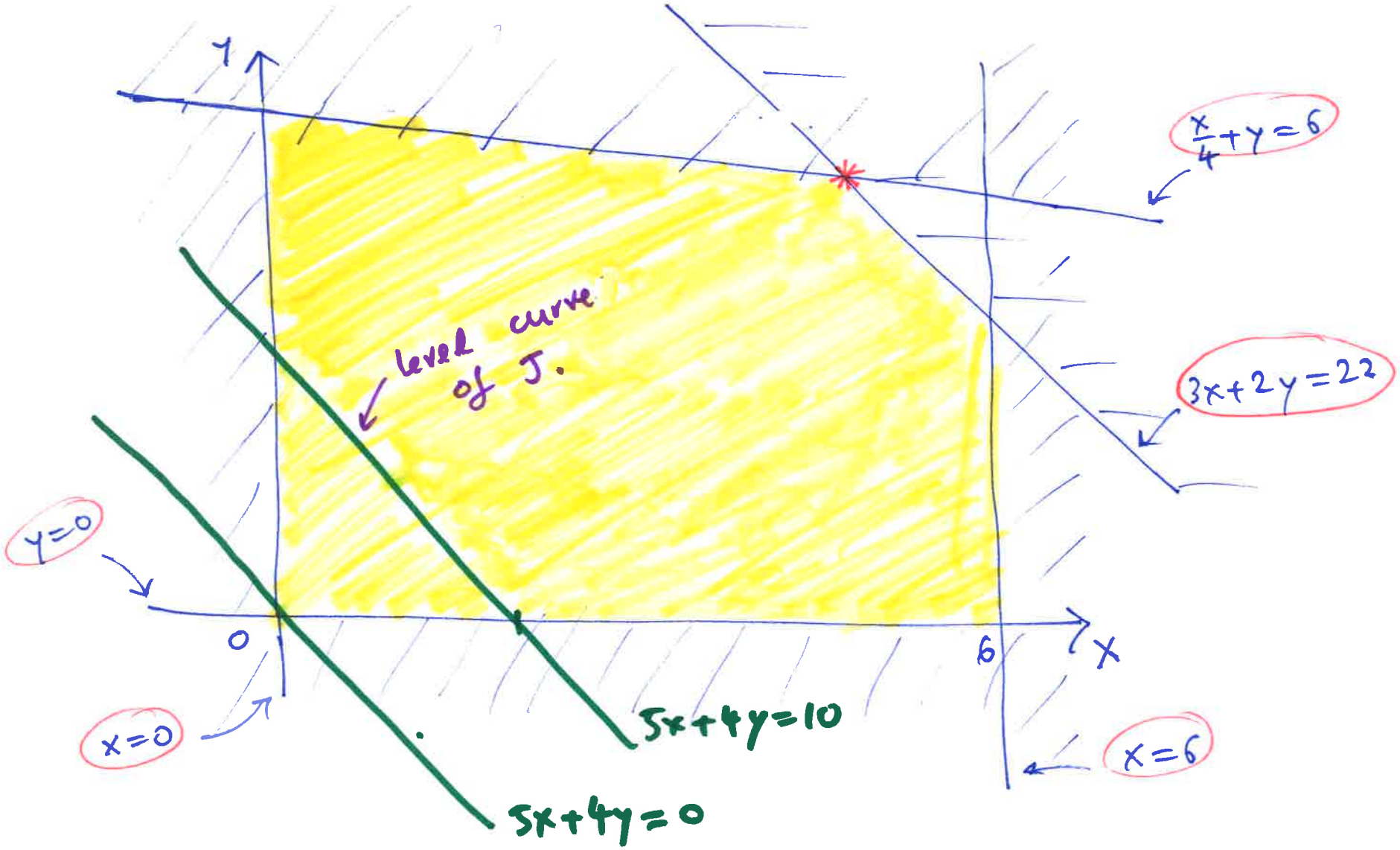
$$3x + 2y \leq 22 \qquad \leftarrow ①,②,③ \quad \text{(resource constraints)}$$

$$x \leq 6 \qquad \leftarrow \cancel{④},⑤ \quad \text{(storage constraints)}$$

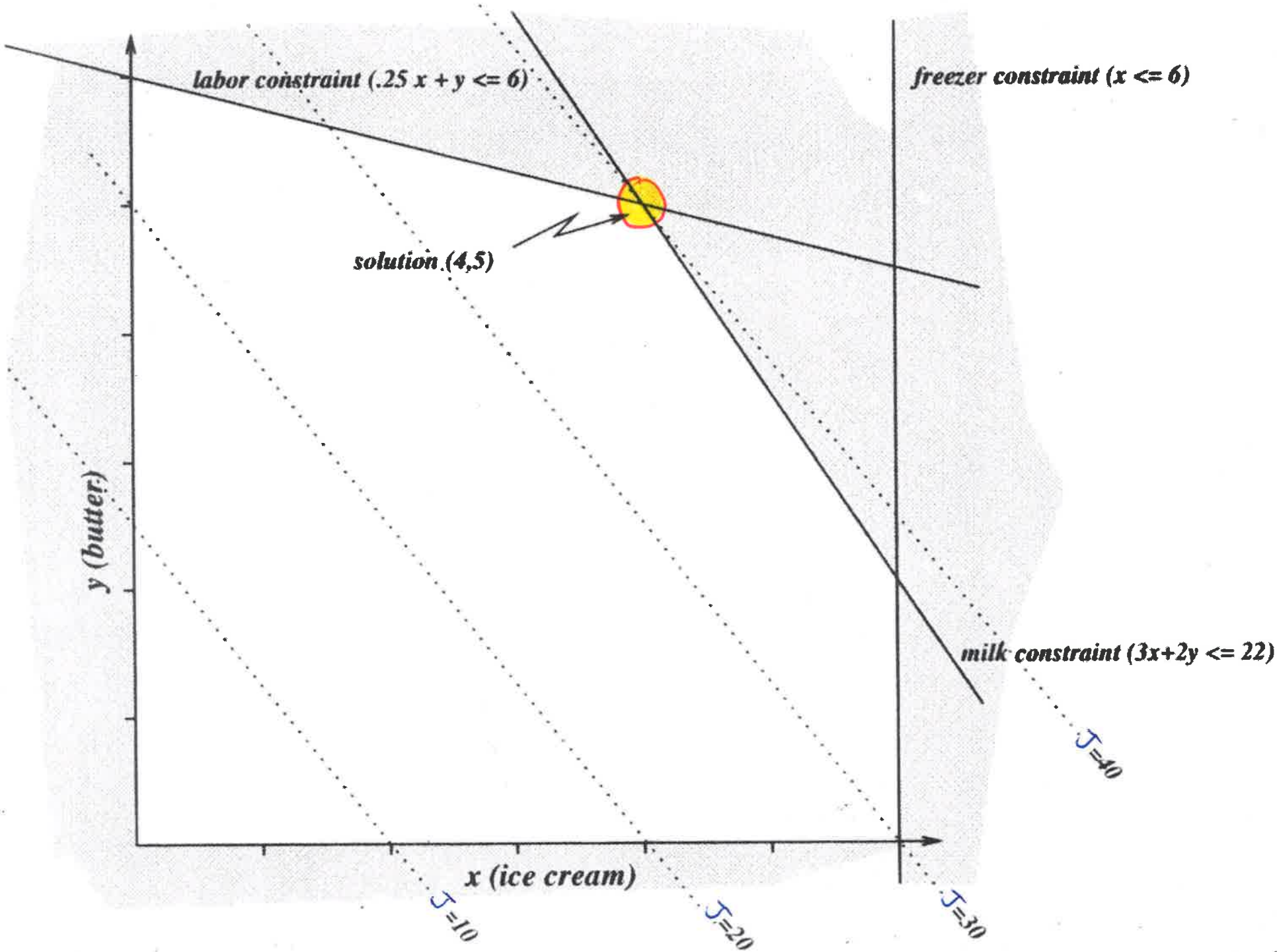
$$\frac{x}{4} + y \leq 6 \qquad \leftarrow ⑥,⑦,⑧ \quad \text{(time constraints)}$$

$$x \geq 0, \; y \geq 0 \qquad \text{(feasibility)}$$

2.1-2



the optimization problem seeks the point in the feasible (yellow) set that maximizes J.


labor constraint (.25 x + y <= 6)
freezer constraint (x <= 6)
solution (4,5)
milk constraint (3x+2y <= 22)
y (butter)
x (ice cream)
J=10
J=20
J=30
J=40

- maximum profit corresponds to the highest level curve that is still (barely) in the feasible set

$$J^* = 40, \quad x^* = 4, \quad y^* = 5.$$ (doesn't have to be integer-valued)

- the solution (almost) always occurs at a corner (vertex)

- the graphical description allows easy answers to questions such as
  - example: how increase/decrease in ice cream & butter prices affect the optimal solution?
    (corresponds to changes in the slope of level curves)
  - example: would be a good idea to buy extra milk at say $1 per gallon?

- graphical method is very difficult if we have more than 2 variables (say we make yogurt & cheese too!)

- the "dairy" problem can be rewritten as

$$\text{minimize} \quad J(x) = c^T z \qquad z = \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\text{subject to} \quad Az \leq b$$

with $c = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ \frac{1}{4} & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 22 \\ 6 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$

this is an example of a "linear program".

- applications of LP include:

  - the diet problem:

    x_j: amount of food j

    F_{ij}: amount of nutrient i in food j

    r_i: minimum daily requirement of nutrient i

    c_j: cost of serving unit amount of food j

minimize $c^T x$

subject to $Fx \geq r$, $x \geq 0$

rewrite to put in standard LP form

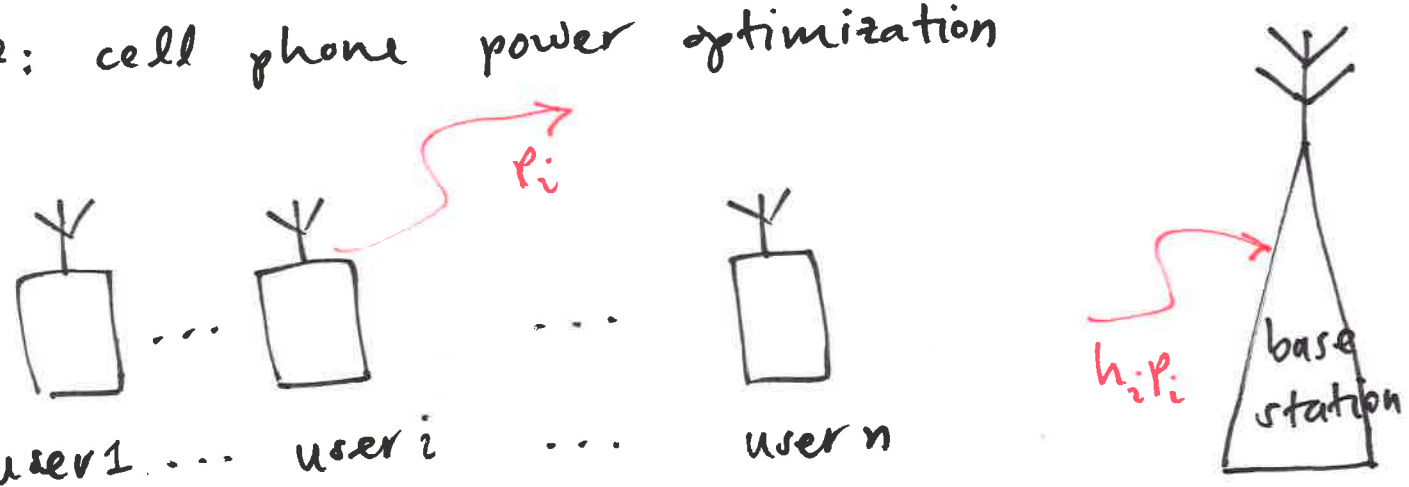
$$-Fx \leq -r$$
$$-x \leq 0$$

$$\begin{bmatrix} -F \\ -I \end{bmatrix} x \leq \begin{bmatrix} -r \\ 0 \end{bmatrix}$$

A $\quad$ b

Kantorovich.

- example: cell phone power optimization



- user i transmits with power p_i
- attenuation factor h_i (for p_i)
  (actual power received at base station for user i is $h_i p_i$)
- for reliable communication, the signal-to-interference ratio for user i must exceed γ_i.
- interference for user i from all other users is $\sum_{j \neq i} h_j p_j$

it is desired to find p_i, $i=1,\dots,n$, that achieve reliable communication, but minimize total power usage.

$$\begin{aligned} \text{minimize} \quad & p_1+\cdots+p_n \\ \text{subject to} \quad & p_i \geq 0 \qquad i=1,\dots,n \\ & \frac{h_i p_i}{\sum_{j\neq i} h_j p_j} \geq \gamma_i \qquad i=1,\dots,n \end{aligned}$$

$$p := \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \qquad c := \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \longrightarrow c^T p = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = p_1+\cdots+p_n .$$

$$p_i \geq 0 \ \ \forall i \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{I} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \geq 0 \longrightarrow \underbrace{-I}_{A_1} p \leq \underbrace{0}_{b_1} .$$

$$h_i p_i \geq \gamma_i \sum_{j \neq i} h_j p_j \quad \rightarrow \quad -h_i p_i + \gamma_i \sum_{j \neq i} h_j p_j \leq 0$$

$$\gamma_i (h_1 p_1 + \cdots + h_{i-1} p_{i-1} + h_{i+1} p_{i+1} + \cdots + h_n p_n)$$

$$\begin{bmatrix} +\gamma_i h_1 & \cdots & +\gamma_i h_{i-1} & -h_i & +\gamma_i h_{i+1} & \cdots & +\gamma_i h_n \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_i \\ \vdots \\ p_n \end{bmatrix} \leq 0$$

$$\begin{bmatrix} -h_1 & \gamma_1 h_2 & \cdots \\ \gamma_2 h_1 & -h_2 & \cdots \\ \vdots & & \ddots \\ \gamma_i h_1 & \cdots & -h_i \\ \vdots & & \\ \gamma_n h_1 & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \gamma_1 h_n \\ \gamma_2 h_n \\ \vdots \\ \gamma_i h_n \\ \vdots \\ -h_n \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_i \\ \vdots \\ p_n \end{bmatrix} \leq 0 \quad \rightarrow A_2 p \leq b_2$$

b_2

A_2

minimize $c^T p$

subject to $A_1 p \leq 0$

$A_2 p \leq 0$

$\Big\} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} p \leq \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

A $\quad$ b

- example: maximize $x_2 - x_1$ $\longrightarrow$ minimize $x_1 - x_2$

subject to $3x_1 = x_2 - 5$

$x_1 \leq 0$

$|x_2| \leq 2$.

$c := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $\quad G := \begin{bmatrix} 3 & -1 \end{bmatrix}$ $\quad h := -5$

$|x_2| \leq 2 \longrightarrow -2 \leq x_2 \leq 2 \longrightarrow \begin{cases} x_2 \leq 2 \\ x_2 \geq -2 \end{cases} \longrightarrow \begin{cases} x_2 \leq 2 \\ -x_2 \leq 2 \end{cases}$

$$\begin{cases} x_1 \le 0 \\ x_2 \le 2 \\ -x_2 \le 2 \end{cases} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \longrightarrow A := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

- example: minimize $c^T x$
  subject to $\|x\|_\infty \le 1$.

$\|x\|_\infty$ = maximum $\{ |x_1|, \cdots, |x_n| \}$

$\downarrow$

$\|x\|_\infty \le 1 \iff$ maximum $\{ |x_1|, \cdots, |x_n| \} \le 1$

$\Updownarrow$

$|x_1| \le 1, \cdots, |x_n| \le 1$

$\Updownarrow$

$-1 \le x_1 \le 1, \cdots, -1 \le x_n \le 1.$

aside:

$\|x\|_\infty = \max_i |x_i|.$

$x = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \rightarrow \|x\|_\infty = 5$

$$\begin{cases} x_1 \le 1 \\ -x_1 \le 1 \\ \vdots \\ x_n \le 1 \\ -x_n \le 1 \end{cases} \longrightarrow A := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots 0 \\ 0 & -1 & 0 & \cdots 0 \\ & & \ddots & \\ & & \ddots & \\ & & & \ddots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & -1 \end{bmatrix} \qquad b := \mathbb{1}.$$

- **example:** minimize $\max_{i=1,\cdots,m} (a_i^T x + b_i)$

$\Updownarrow$ equivalent epigraph form

minimize t

subject to $\max_i (a_i^T x + b_i) \le t$

$\Updownarrow$

minimize t

subject to $a_i^T x + b_i \le t, \quad i = 1, \cdots, m$

aside: note that in problems of the form $\underset{x\in X}{\text{minimize}} \sup_{y\in Y} f(x,y)$ that are written in epigraph form

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & \sup_{y\in Y} f(x,y) \le t \\ & x \in X \end{aligned}$$

to eliminate the "sup" the inequalities $f(x,y) \le t$ have to be enforced for every y. the above problem is not equivalent to

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & f(x,y) \le t \\ & y \in Y \\ & x \in X \end{aligned}$$

for example, consider $\underset{x\ge 1}{\text{minimize}} \sup_{0\le y\le 1} x^2 y^2$, which has the solution $x^* = 1$, $p^* = 1$

this problem is not equivalent to

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & x^2 y^2 \le t \\ & 0 \le y \le 1 \\ & x \ge 1 \end{aligned}$$

(which has the solution $p^* = 0$)

as another example consider

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\dots,k} v_i^T X v_i \\ X \in \mathcal{X} & \end{array} \iff \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & v_i^T X v_i \le t \quad i=1,\dots,k \\ & X \in \mathcal{X} \end{array}$$

on the other hand

$$\begin{array}{ll} \text{minimize} & \sup_{\|v\|_2 \le 1} v^T X v \\ X \in \mathcal{X} & \end{array} \not\iff \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & v^T X v \le t \\ & v^T v \le 1 \\ & X \in \mathcal{X} \end{array}$$

which has solution $\begin{cases} p^* = 0 \\ v^* = 0 \\ t^* = 0 \end{cases}$

$$\Updownarrow$$

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(X) \\ X \in \mathcal{X} & \end{array}$$

$$\Updownarrow$$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & X \preceq tI \end{array}$$

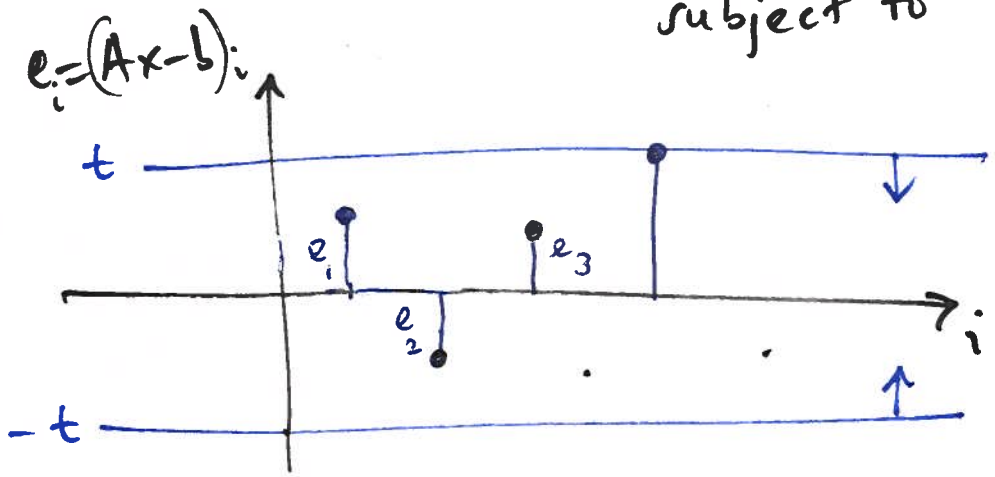
- example: minimize $\|Ax-b\|_\infty$

aside: $\|e\|_\infty = \max_i |e_i|$.

$\Updownarrow$

minimize t

subject to $\|Ax-b\|_\infty \le t$.

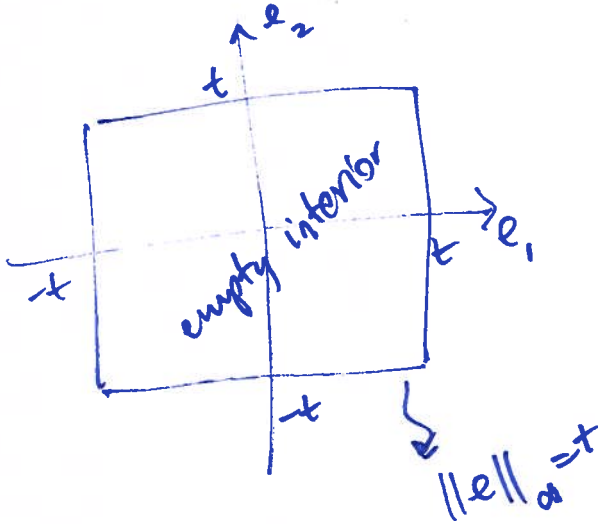


$$\|Ax-b\|_\infty \le t \iff \text{maximum}\{\underbrace{|(Ax-b)_1|}_{e_1}, \ldots, \underbrace{|(Ax-b)_m|}_{e_m}\} \le t$$

$$\iff |(Ax-b)_1| \le t, \ldots, |(Ax-b)_m| \le t$$

$$\iff -t \le (Ax-b)_1 \le t, \ldots, -t \le (Ax-b)_m \le t$$

$$\iff -t\mathbb{1} \le Ax-b \le t\mathbb{1}$$



new vector of optimization variables $\hat{x} = \begin{bmatrix} x \\ t \end{bmatrix}$.

$c := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ← zero is of same dim. as x.

minimize $\begin{bmatrix} 0^T & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$

$\begin{bmatrix} x \\ t \end{bmatrix}$

subject to $\begin{bmatrix} A & -\mathbb{1} \\ -A & -\mathbb{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix}$

• example:

$$\begin{aligned} &\text{minimize} && \|x\|_\infty \\ &\text{subject to} && Ax = b. \end{aligned}$$

↓

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && \|x\|_\infty \leq t \\ & && Ax = b \end{aligned}$$

relaxed from equality (which will always be satisfied)

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && -t\mathbb{1} \leq x \leq t\mathbb{1} \\ & && Ax = b \end{aligned}$$

↓

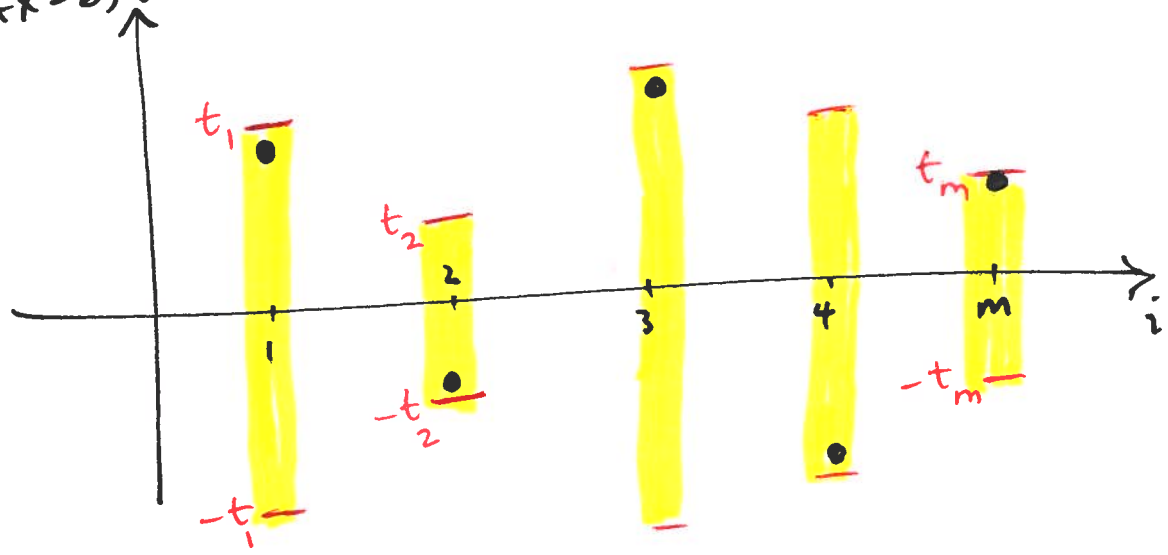
$$x - t\mathbb{1} \leq 0$$
$$-x - t\mathbb{1} \leq 0$$

↓

$$\begin{bmatrix} +I & -\mathbb{1} \\ -I & -\mathbb{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- example: minimize $\|Ax-b\|_1$

$e_i = (Ax-b)_i$



$$\|Ax-b\|_1 = \underbrace{|(Ax-b)_1|}_{t_1} + \cdots + \underbrace{|(Ax-b)_m|}_{t_m}$$

$$\text{min. } \|Ax-b\|_1 \quad \leftrightarrow \quad \begin{array}{ll} \text{min.} & t_1+\cdots+t_m \\ \text{s.t.} & |(Ax-b)_i| \le t_i \end{array} \qquad i=1,\cdots,m$$

aside 1:

$\|e\|_1 = |e_1| + \cdots + |e_m|$

aside 2:

if $Ax-b \geq 0$

$\downarrow$

$\|Ax-b\|_1 = \mathbb{1}^T(Ax-b)$

$= \underbrace{(\mathbb{1}^TA)}_{c^T} x - \underbrace{\mathbb{1}^Tb}_{d}$

$$\begin{aligned} \min. \quad & t_1 + \cdots + t_m \\ \text{s.t.} \quad & |(Ax-b)_i| \le t_i \qquad i = 1, \cdots, m \end{aligned}$$

$$|(Ax-b)_i| \le t_i \iff -t_i \le (Ax-b)_i \le t_i$$

$$\iff -\begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix} \le \begin{bmatrix} (Ax-b)_1 \\ \vdots \\ (Ax-b)_m \end{bmatrix} \le \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix}$$

$$\iff -t \le Ax-b \le t.$$

$\updownarrow$

$\updownarrow$

minimize $\begin{bmatrix} x \\ t \end{bmatrix}$ $\quad \mathbb{1}^T t$

subject to $\quad -t \leq Ax - b \leq t$

$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+m}$

$\downarrow$

minimize $\quad \begin{bmatrix} 0^T & \mathbb{1}^T \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$

subject to $\quad \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix}.$

aside: Compare problem on top of page with the "$\min \|Ax-b\|_\infty$" problem:

minimize $\quad t$

subject to $\quad -t\,\mathbb{1} \leq Ax - b \leq t\,\mathbb{1}.$

- example: minimize $\|x\|_1$
  subject to $Ax = b$

} ~ compressed sensing

aside: if x is "sparse" in the sense that it has a lot of zeros, then A can be very fat & you can still find the actual x

(in this scenario, x would be an actual unknown and A, b would represent the measurements of x.)



minimize $t_1 + \cdots + t_n$

subject to $|x_i| \leq t_i \quad i = 1, \ldots, n$

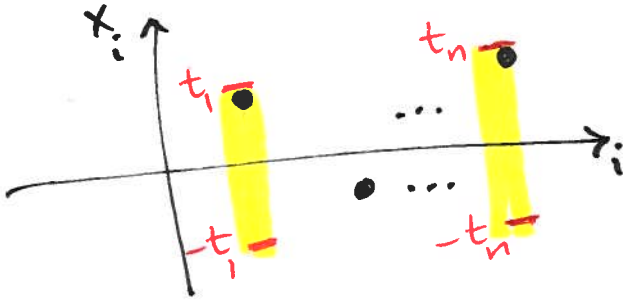
$Ax = b$



minimize $\mathbb{1}^T t$

subject to $-t \leq x \leq t$

$Ax = b.$



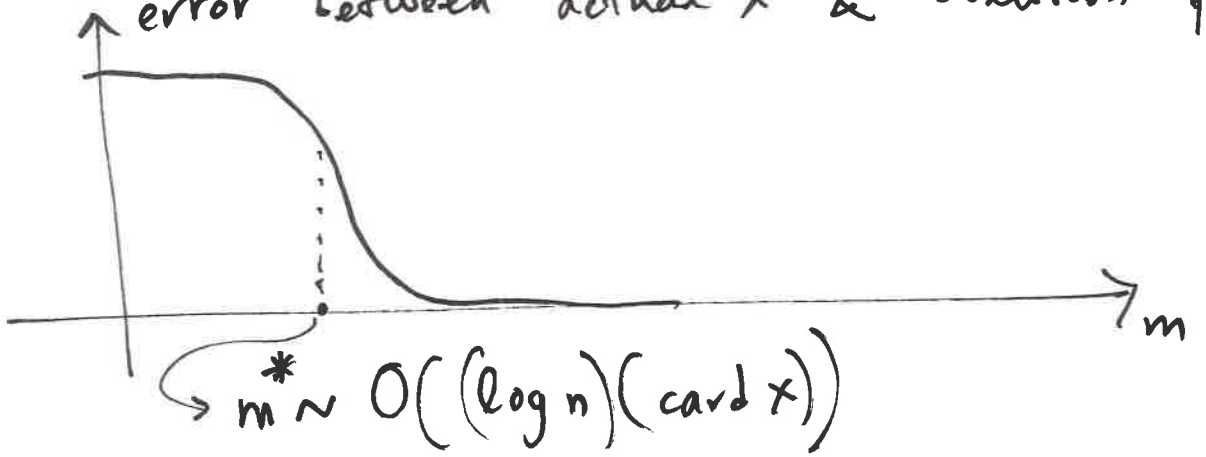
- compressive/compressed sensing

minimize $\|x\|_1 = \sum_i |x_i|$

subject to $Ax = y$

$x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $m \leq n$

(interesting when A very fat)



$n = 10^6$, $\text{card } x \approx 10^3$ $\longrightarrow$ $m \sim O(6000)$

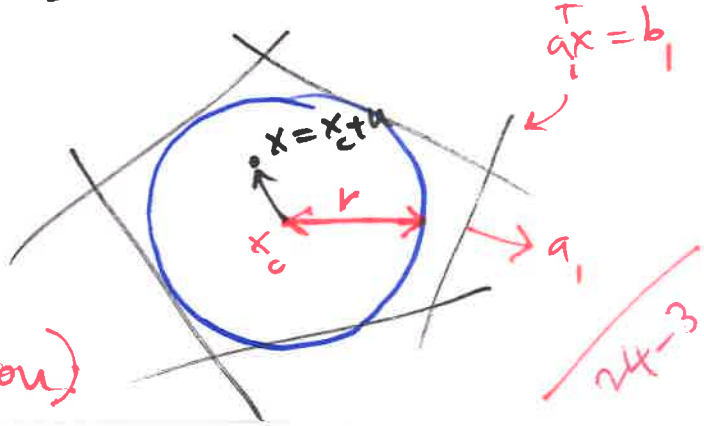
- Chebyshev center of a polyhedron

this is defined as the point in the polyhedron

$P = \{x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m\}$ which is the center of the largest inscribed ball,

$B(x_c, r) = \{x_c + u \mid \|u\|_2 \leq r\}$.

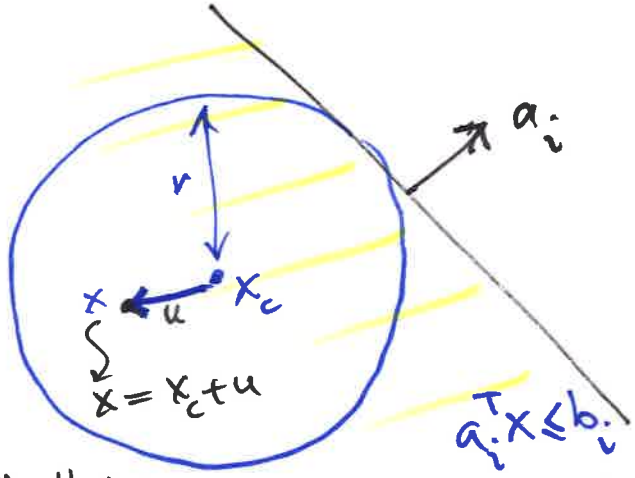
(x_c: "deepest" point inside the polyhedron)



- example: given A, b, find x_c and r.

objective: maximize r

constraint: ball is inside the polyhedron



i^{th} constraint: $a_i^T x \le b_i$ for all $x \in B(x_c, r)$

$\Updownarrow$

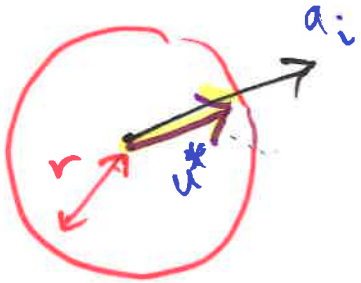
$a_i^T(x_c + u) \le b_i$ for all u with $\|u\|_2 \le r$

$\Updownarrow$

$$\left\{ \begin{array}{l} \text{maximum} \\ \|u\|_2 \le r \end{array} \; a_i^T(x_c + u) \right\} \le b_i$$

$$a_i^T x_c + \left\{ \begin{array}{l} \text{maximum} \\ \|u\|_2 \le r \end{array} \quad a_i^T u \right\} \le b_i$$

$$u^* = r \frac{a_i}{\|a_i\|_2}$$

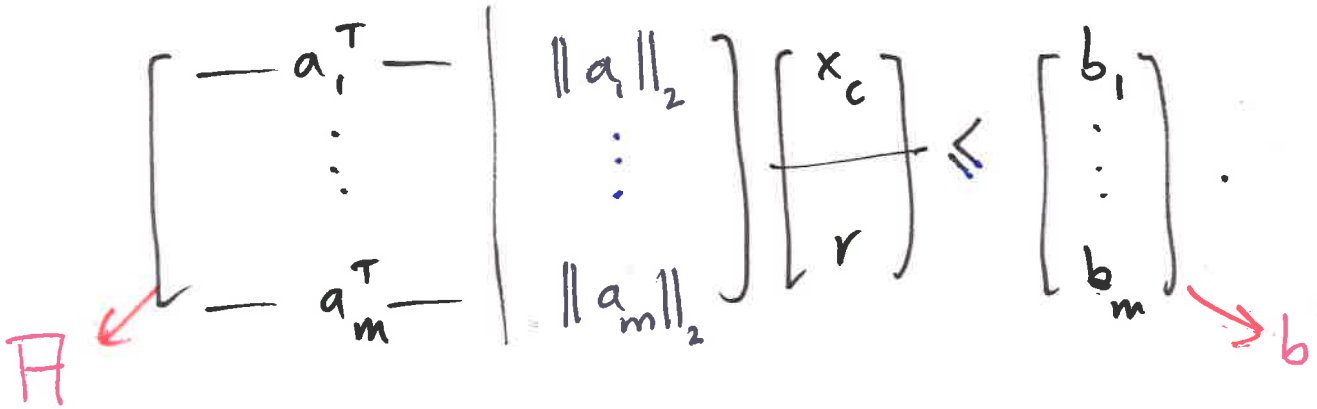


$$\left(a_i^T u = \|a_i\| \, \|u\| \cos\theta \right)$$

$$a_i^T x_c + a_i^T \left(r \frac{a_i}{\|a_i\|_2} \right) \le b_i$$

$$a_i^T x_c + \|a_i\|_2 \, r \le b_i$$

now, repeat this procedure for every $i = 1, \cdots, m$



the formulation of the optimization as an LP is given by

$$\begin{aligned} \text{maximize} \quad & r \\ \text{subject to} \quad & A\begin{bmatrix} x_c \\ r \end{bmatrix} \leq b \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \text{minimize} \quad & -\begin{bmatrix} 0^T & 1 \end{bmatrix}\begin{bmatrix} x_c \\ r \end{bmatrix} \\ \text{subject to} \quad & A\begin{bmatrix} x_c \\ r \end{bmatrix} \leq b \end{aligned}$$

- example: consider a probability distribution for a discrete random variable x on a set $\{u_1, \ldots, u_n\} \subseteq \mathbb{R}$

we describe the distribution of x by $p \in \mathbb{R}^n$, with

$p_i = \text{prob}(x = u_i)$

$p \succeq 0 \quad \mathbb{1}^T p = 1$



We assume u_i are known and p is unknown.

if f is any function of x, then

$$E\{f\} = \sum_{i=1}^{n} p_i f(u_i)$$

is a linear function of p. if $S \subseteq \mathbb{R}$, then

$$\text{prob}(x \in S) = \sum_{u_i \in S} p_i$$

is a linear function of p.

suppose, we know upper & lower bounds on expected values of certain functions of x

$$\alpha_i \le a_i^T p \le \beta_i, \quad i=1,\dots,m.$$

the problem is to give lower & upper bounds on $\mathcal{E}\{f_0\} = \sum_i p_i f_0(u_i) = a_0^T p$ (where f_0 is some function of x.)

to find lower bound:

$$\begin{aligned} &\text{minimize} && a_0^T p \\ &\text{subject to} && p \succeq 0, \quad \mathbf{1}^T p = 1 \\ & && \alpha_i \le a_i^T p \le \beta_i \quad i=1,\dots,m \end{aligned}$$

to find upper bound:

$$\begin{aligned} &\text{maximize} && a_0^T p \\ &\text{subject to} && \text{same constraints as above} \end{aligned}$$

recall

- example: consider the linear-fractional program

$$\begin{aligned} \text{minimize} \quad & f_0(x) = \frac{c^T x + d}{e^T x + f} \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned} \qquad (1)$$

$$\text{dom } f_0 = \{ x \mid e^T x + f > 0 \}.$$

the objective is quasiconvex (and quasiconcave, hence quasilinear). this problem can be transformed to an equivalent linear program,

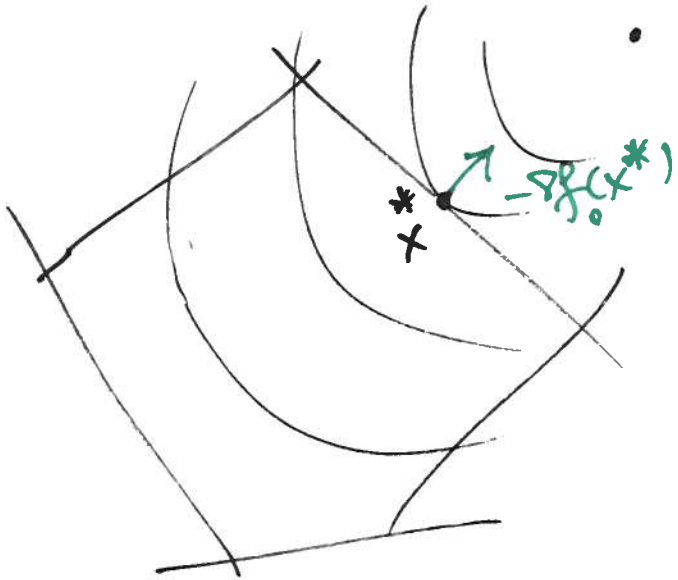
$$\begin{aligned} \text{minimize} \quad & c^T y + dz \\ \text{subject to} \quad & Gy - hz \preceq 0 \\ & Ay - bz = 0 \\ & e^T y + fz = 1 \\ & z \geq 0 \end{aligned} \qquad (2)$$

- quadratic optimization problems:

  - a convex optimization problem is called a quadratic program (QP) if it can expressed as

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}x^T P x + q^T x + r \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

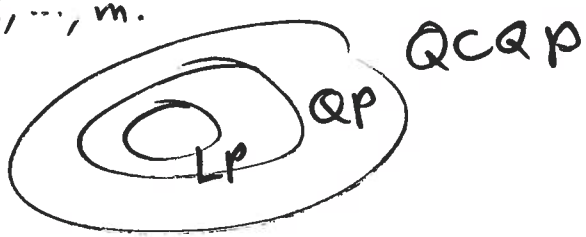
where $P \in S_+^n$, $G \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{p \times n}$



– a convex optimization problem is called a quadratically constrained quadratic program (QCQP) if it can be expressed as

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} \quad & \tfrac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0 \qquad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- quadratic programs include linear programs as a special case by taking $P = 0$. quadratically constrained quadratic programs include quadratic programs as a special case by taking $P_i = 0$, $i = 1, \ldots, m$.



- example: Euclidean distance between polyhedra defined as

$$P_1 = \{x \mid A_1 x \preceq b_1\}, \text{ and } P_2 = \{x \mid A_2 x \preceq b_2\} \text{ in } \mathbb{R}^n$$

$$\text{dist}(P_1, P_2) = \inf \{ \|x_1 - x_2\|_2 \mid x_1 \in P_1, \; x_2 \in P_2 \}$$

to find the distance between P_1 & P_2 we solve

$$\begin{array}{ll} \text{minimize} & \|x_1 - x_2\|_2^2 \\ \text{subject to} & A_1 x_1 \preceq b_1 \\ & A_2 x_2 \preceq b_2 \end{array}$$

- example: the least squares problem is an unconstrained QP

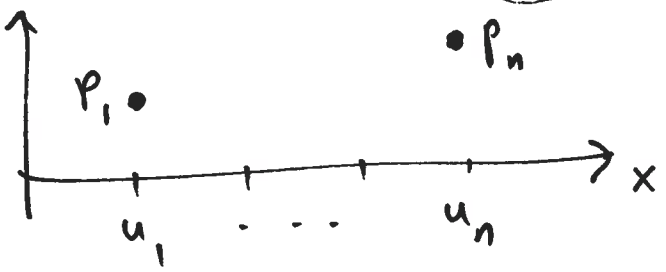
$$\|Ax - b\|_2^2 = x^T A^T A x - 2 b^T A x + b b^T$$

- example: consider a problem where the variable is an unknown probability distribution given by $p \in \mathbb{R}^n$. the variance of a random variable $f(x)$ is given by

$$\mathcal{E}\{(f - \underbrace{\mathcal{E}\{f\}}_{\bar{f}})^2\} = \mathcal{E}\{f^2 - 2f\bar{f} + \bar{f}^2\} = \mathcal{E}\{f^2\} - 2\mathcal{E}\{f\}\bar{f} + \bar{f}^2$$

$$= \mathcal{E}\{f^2\} - (\mathcal{E}\{f\})^2 = \underbrace{\sum_{i=1}^n f_i^2 p_i - \left(\sum_{i=1}^n f_i p_i\right)^2}_{\text{concave quadratic}}$$

where $f_i = f(u_i)$.



we can maximize the variance of $f(x)$ subject to prior information (in the form of bounds on $a_i^T p$)

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n f_i^2 p_i - \left(\sum_{i=1}^n f_i p_i\right)^2 \\ \text{subject to} & p \succeq 0, \quad \mathbb{1}^T p = 1 \\ & \alpha_i \le a_i^T p \le \beta_i \quad i = 1, \ldots, m \end{array}$$

- recall the diet problem:

x_j: amount of food j

F_{ij}: amount of nutrient i in food j

r_i: minimum daily requirement of nutrient i

c_j: cost of serving unit amount of food j

minimize $c^T x$

subject to $Fx \geq r$, $x \geq 0$

rewrite to put in standard LP form →

$-Fx \leq -r$

$-x \leq 0$

$$\begin{bmatrix} -F \\ -I \end{bmatrix} x \leq \begin{bmatrix} -r \\ 0 \end{bmatrix}$$

A $\quad$ b

Kantorovich.

- example: diet problem with uncertainty

$$\text{minimize} \quad \mathcal{E}\{c^T x\} + \gamma \, \text{var}\{c^T x\}$$
$$\text{subject to} \quad Fx \succcurlyeq r, \quad Ax = b, \quad x \succcurlyeq 0$$

with $\mathcal{E}\{c\} = \bar{c}$, $\mathcal{E}\{(c-\bar{c})(c-\bar{c})^T\} = \Sigma$

c is a random variable with mean $\bar{c}$ & variance Σ.

$$\text{var}(c^T x) = \mathcal{E}\{(c^T x - \bar{c}^T x)(\cdots)^T\} = \mathcal{E}\{x^T (c-\bar{c})(c-\bar{c})^T x\}$$

$$= x^T \Sigma x .$$

γ is the risk-aversion parameter

$$\text{minimize} \quad \bar{c}^T x + \gamma x^T \Sigma x$$
$$\text{subject to} \quad Fx \succcurlyeq r, \quad Ax = b, \quad x \succcurlyeq 0 .$$

aside: consider the problem

$$\begin{array}{ll} \text{minimize} & x^T A_0 x + 2b_0^T x + c_0 \\ \text{subject to} & x^T A_1 x + 2b_1^T x + c_1 \leq 0 \end{array}$$

we do not assume that $A_i \succeq 0$ so this is not a convex optimization problem.

$$\begin{aligned} x^T A_i x &= \text{trace}(x^T A_i x) \\ &= \text{trace}(A_i x x^T) \\ &= \text{trace}(A_i X) \qquad X = xx^T \end{aligned}$$

so the original optimization problem is equivalent to

$$\begin{aligned} \text{minimize} \quad & \operatorname{trace}(A_0 X) + 2b_0^T x + c_0 \\ \text{subject to} \quad & \operatorname{trace}(A_1 X) + 2b_1^T x + c_1 \leq 0 \end{aligned}$$

$$X = xx^T$$

replacing $X = xx^T$ with $X \succcurlyeq xx^T$ we get

$$\left.\begin{aligned} \text{mininimize} \quad & \cdots \\ \text{subject to} \quad & \cdots \\ & X \succcurlyeq xx^T \end{aligned}\right\} \text{semidefinite program (SDP)}$$

$$X - xx^T \succcurlyeq 0 \iff \underbrace{\begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succcurlyeq 0}_{\text{LMI}}$$

the optimal value of the last problem is less than or equal to that of the original problem. if $X = xx^T$ at the optimum of the last problem then x is the optimal of the original problem.

- second-order cone programming: a problem closely related to quadratic programing (but more general than QCQP) is second-order cone program (SOCP)

$$\begin{aligned} \text{minimize} \quad & f^T x \\ \text{subject to} \quad & \|A_i x + b_i\|_2 \leq c_i^T x + d_i \qquad i = 1, \dots, m \\ & Fx = g \end{aligned}$$

where $A_i \in \mathbb{R}^{n_i \times n}$, $F \in \mathbb{R}^{p \times n}$.

when $c_i = 0$, $i = 1, \dots, m$ the SOCP is equivalent to a QCQP. if $A_i = 0$, $i = 1, \dots, m$ the the SOCP is equivalent to an LP.

- example: consider a linear program

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && a_i^T x \le b_i \qquad i = 1, \dots, m \end{aligned}$$

in which there is uncertainty/variation in the parameters c, a_i, b_i. here, assume c & b_i fixed and the a_i are known to lie in given ellipsoids

$$a_i \in \mathcal{E}_i = \{ \bar{a}_i + P_i u \mid \|u\|_2 \le 1 \}$$

requiring that the constraints be satisfied for all possible values of a_i, we get

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && a_i^T x \le b_i \quad \text{for all} \quad a_i \in \mathcal{E}_i \\ & && \qquad\qquad i = 1, \dots, m \end{aligned}$$

$$a_i^T x \le b_i \quad \text{for all} \quad a_i \in \mathcal{E}_i$$

$$\updownarrow$$

$$\sup \{ a_i^T x \mid a_i \in \mathcal{E}_i \} \le b_i$$

$$\sup \{ a_i^T x \mid a_i \in \mathcal{E}_i \} = \sup \{ (\bar{a}_i + P_i u)^T x \mid \|u\|_2 \le 1 \}$$

$$= \bar{a}_i^T x + \sup \{ u^T P_i^T x \mid \|u\|_2 \le 1 \}$$

$$u^* = \frac{P_i^T x}{\|P_i^T x\|_2} \longleftarrow (=) \quad \bar{a}_i^T x + \|P_i^T x\|_2$$

---

the problem becomes

$$\text{minimize} \quad c^T x$$

$$\text{subject to} \quad \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i \qquad i = 1, \ldots, m$$

- example: robust LP can also be considered in a statistical framework. suppose that a_i are independent Gaussian random vectors

$$\mathcal{E}\{a_i\} = \bar{a}_i$$

$$\mathcal{E}\{(a_i - \bar{a}_i)(a_i - \bar{a}_i)^T\} = \Sigma_i$$

we require that

$$\text{prob}(a_i^T x \leq b_i) \geq \eta$$

letting $u = a_i^T x$ with mean μ and variance σ^2 the constraint above becomes

$$\text{prob}\left(\frac{u-\mu}{\sigma} \leq \frac{b_i-\mu}{\sigma}\right) \geq \eta$$

$$\updownarrow \quad \frac{u-\mu}{\sigma} \sim N(0,1)$$

$$\Phi\left(\frac{b_i-\mu}{\sigma}\right) \geq \eta \qquad \text{where} \qquad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

cumulative distribution of $N(0,1)$

$$\updownarrow$$

$$\frac{b_i-\mu}{\sigma} \geq \Phi^{-1}(\eta)$$

$$\updownarrow$$

$$\mu + \Phi^{-1}(\eta)\,\sigma \leq b_i$$

since

$$\mu = \mathcal{E}\{a_i^T x\} = \bar{a}_i^T x$$

$$\sigma^2 = \mathcal{E}\{(a_i^T x - \bar{a}_i^T x)^2\}$$

$$= x^T \mathcal{E}\{(a_i - \bar{a}_i)(a_i - \bar{a}_i)^T\} x = x^T \Sigma_i x$$

$$\sigma = (x^T \Sigma_i x)^{1/2}$$

we get

$$\mu + \Phi^{-1}(\eta)\,\sigma \leq b_i$$

$$\updownarrow$$

$$\bar{a}_i^T x + \Phi^{-1}(\eta)\,\|\Sigma_i^{1/2} x\|_2 \leq b_i$$

---

aside: recall that SOCP constraints are of the form $\|A_i x + b_i\|_2 \leq c_i^T x + d_i$

if $\eta \geq \frac{1}{2}$ then $\Phi^{-1}(\eta) \geq 0$ so this constraint is a second-order cone constraint.

the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \text{prob}(a_i^T x \leq b_i) \geq \eta \quad i=1,\ldots,m \end{array}$$

can be expressed as the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \left\| \Sigma_i^{1/2} x \right\|_2 \leq b_i \\ & i=1,\ldots,m \end{array}$$

(compare to the result when $a_i \in \mathcal{E}_i = \{ \bar{a}_i + P_i u \mid \|u\|_2 \leq 1 \}$

which lead to

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i \quad i=1,\ldots,m) \end{array}$$

- example: consider a differentiable function $f: \mathbb{R}^2 \to \mathbb{R}$, with $\operatorname{dom} f = C$. the surface area of its graph is given by

$$A = \int_C \sqrt{1 + \|\nabla f(x)\|_2^2}\, dx$$

$$= \iint_C \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2}\, dx_1 dx_2$$

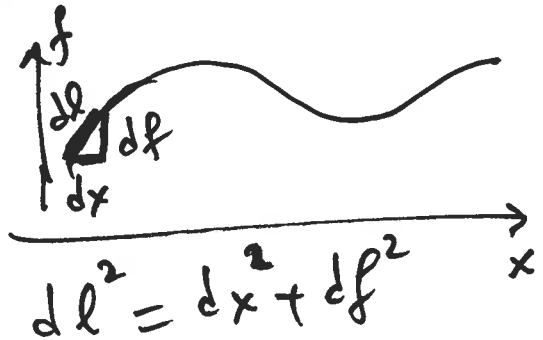
$$= \int_C \left\| \left(\nabla f(x), 1\right) \right\|_2 dx$$

we wish to solve a minimal surface problem.

---

aside:

in 1-dim



$$dl^2 = dx^2 + df^2$$

$$\int \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = \int \sqrt{dx^2 + df^2}$$

$$= \int dl$$

Let $C = [0,1]^2$, and $f_{ij} \sim f\left(\frac{i}{K}, \frac{j}{K}\right)$, $i,j = 0, \ldots, K$.

$$\nabla f(x) \approx \begin{bmatrix} \dfrac{f_{i+1,j} - f_{ij}}{\frac{i+1}{K} - \frac{i}{K}} \\ \dfrac{f_{i,j+1} - f_{ij}}{\frac{j+1}{K} - \frac{j}{K}} \end{bmatrix} = K \begin{bmatrix} f_{i+1,j} - f_{ij} \\ f_{i,j+1} - f_{ij} \end{bmatrix}.$$

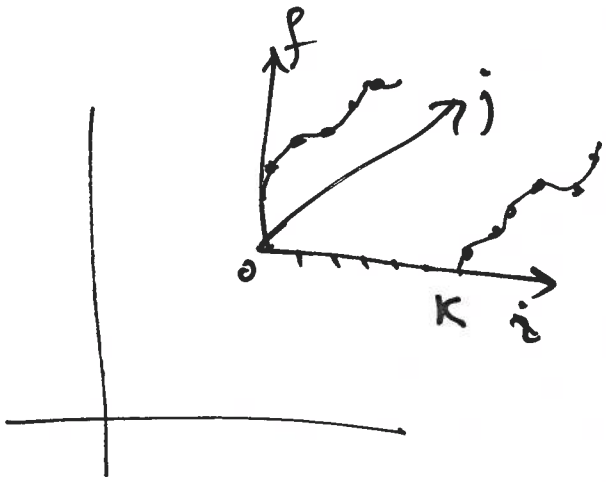
$$A \approx A_{approx} = \sum_{i,j=0}^{K-1} \left\| \begin{bmatrix} K(f_{i+1,j} - f_{ij}) \\ K(f_{i,j+1} - f_{ij}) \\ 1 \end{bmatrix} \right\|_2 \frac{1}{K} \frac{1}{K}$$

an approximation of integral on prev. slide

for instance, we consider the problem of finding the minimal area surface with fixed boundary values on the left & right edges of the square

$$\begin{aligned} &\text{minimize} && A_{approx} \\ &\text{subject to} && f_{0j} = l_j && j=0,\cdots,K \\ &&& f_{Kj} = r_j && j=0,\cdots,K \end{aligned}$$

where f_{ij} are the variables and l_j, r_j are given.



$$\text{minimize} \quad \frac{1}{K^2} \sum_{i,j=0}^{K-1} t_{ij}$$

$$\text{subject to} \quad \left\| \begin{bmatrix} K(f_{i+1,j} - f_{ij}) \\ K(f_{i,j+1} - f_{ij}) \end{bmatrix} \right\|_2 \leq t_{ij} \qquad i,j = 0, \ldots, K-1$$

$$f_{0j} = \ell_j \qquad j = 0, \ldots, K$$

$$f_{Kj} = r_j \qquad j = 0, \ldots, K.$$

aside: suppose f is a symmetric matrix valued function, $f: \mathbb{R}^n \to S^m$. the function f is matrix convex if

$$f(\theta x + (1-\theta) y) \preccurlyeq \theta f(x) + (1-\theta) f(y)$$

for any x, y and $\theta \in [0,1]$. an equivalent definition is that the scalar function $z^T f(x) z$ is convex for all vectors z.

a matrix function is strictly matrix convex if

$$f(\theta x + (1-\theta) y) \prec \theta f(x) + (1-\theta) f(y)$$

when $x \neq y$ an $0 < \theta < 1$. or equivalently, that $z^T f(x) z$ be strictly convex for every $z \neq 0$.

- for example $f(X) = XX^T$, $X \in \mathbb{R}^{n \times m}$ is matrix convex, because for any z, the function $z^T(XX^T)z = \|X^T z\|_2^2$ is a convex quadratic function of the components of X

- as another example, the quadratic matrix function $g(X) = X^T A X + B^T X + X^T B + C$, where $A \in S^m$, $B \in \mathbb{R}^{m \times n}$, $C \in S^n$, is convex when $A \succeq 0$.

- semidefinite programming: an SDP has the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \quad \leftarrow \text{LMI (linear matrix inequality)} \\ & Ax = b \end{array}$$

where $G, F_1, \ldots, F_n \in S^k$ and $A \in \mathbb{R}^{p \times n}$.

- if $G, F_1, \ldots, F_n$ are all diagonal, then the LMI above simplifies to a set of k linear inequalities and the SDP becomes an LP

$$x_1 \underbrace{\begin{bmatrix} f_{11} & & \\ & f_{21} & \\ & & \ddots \\ & & & f_{k1} \end{bmatrix}}_{F_1} + \cdots + x_n \underbrace{\begin{bmatrix} f_{1n} & & \\ & f_{2n} & \\ & & \ddots \\ & & & f_{kn} \end{bmatrix}}_{F_n} + \underbrace{\begin{bmatrix} g_1 & & \\ & g_2 & \\ & & \ddots \\ & & & g_k \end{bmatrix}}_{G} \preceq 0$$

$$\updownarrow$$

$$\begin{bmatrix} f_{11}x_1 + \cdots + f_{1n}x_n \\ \vdots \\ f_{k1}x_1 + \cdots + f_{kn}x_n \end{bmatrix} \preceq \begin{bmatrix} -g_1 \\ \vdots \\ -g_k \end{bmatrix} .$$

(of course, one would never turn an LP into an SDP because LPs can be solved much more efficiently than SDPs can)

– the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F^{(i)}(x) = x_1 F_1^{(i)} + \cdots + x_n F_n^{(i)} + G^{(i)} \preceq 0 \qquad i = 1, \ldots, K \\ & Gx \preceq h \\ & Ax = b \end{array}$$

is transformed to an SDP by forming a large block diagonal LMI

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \operatorname{diag}(Gx - h, F^{(1)}(x), \ldots, F^{(K)}(x)) \preceq 0 \\ & Ax = b \end{array}$$

$$\left(\operatorname{diag}(Gx-h) = \operatorname{diag} \begin{bmatrix} g_{11}x_1 + \cdots + g_{1n}x_n - h_1 \\ \vdots \\ g_{m1}x_1 + \cdots + g_{mn}x_n - h_m \end{bmatrix} = x_1 \begin{bmatrix} g_{11} & & \\ & \ddots & \\ & & g_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} g_{1n} & & \\ & \ddots & \\ & & g_{mn} \end{bmatrix} - \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_m \end{bmatrix} \right)$$

- example: let $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$, where $A_i \in \mathbb{R}^{p \times q}$. consider

$$\text{minimize} \quad \|A(x)\|_2$$

where $\|\cdot\|_2$ is the spectral norm (largest singular value) and $x \in \mathbb{R}^n$ is the optimization variable.

using the fact that $\|A\|_2 \leq s$ if and only if $A^T A \preceq s^2 I$ we can rewrite the above problem as

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & A(x)^T A(x) \preceq s^2 I \end{array}$$

with variables x and s.

the problem is convex because $(A(x))^T(A(x)) - s^2 I \leq 0$

can be written as an LMI

$$A^T A \leq t^2 I \quad (t \geq 0) \iff \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0$$

which gives the SDP

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && \begin{bmatrix} tI & A(x) \\ (A(x))^T & tI \end{bmatrix} \succeq 0 \end{aligned}$$

with variables x and t.

- example: let t be a random variable in $\mathbb{R}$. the expected values $\mathcal{E}\{t^k\}$ are called the moments of the distribution of t. if there is a probability distribution on $\mathbb{R}$ such that $x_k = \mathcal{E}\{t^k\}$, $k=0,\dots,2n$, then we must have that $x_0 = 1$ and

$$H(x_0,\dots,x_{2n}) = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_1 & x_2 & x_3 & \cdots & x_n & x_{n+1} \\ x_2 & x_3 & x_4 & \cdots & x_{n+1} & x_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-1} & x_n & x_{n+1} & \cdots & x_{2n-2} & x_{2n-1} \\ x_n & x_{n+1} & x_{n+2} & \cdots & x_{2n-1} & x_{2n} \end{bmatrix} \geq 0$$

now suppose t is a random variable on $\mathbb{R}$ and we do not know its distribution, but we do know some bounds on the moments

$$\underline{\mu}_k \leq \mathcal{E}\{t^k\} \leq \bar{\mu}_k \,, \quad k=1,\dots,2n.$$

let $p(t) = c_0 + c_1 t + \cdots + c_{2n} t^{2n}$ be a given polynomial. then

$$\mathcal{E}\{p(t)\} = \sum_{i=0}^{2n} c_i \, \mathcal{E}\{t^i\} = \sum_{i=0}^{2n} c_i x_i$$

we can compute upper / lower bounds for $\mathcal{E}\{p(t)\}$ over all probability distributions that satisfy the given moment bounds by solving the SDP

$$\begin{array}{ll} \text{minimize/maximize} & c_1 x_1 + \cdots + c_{2n} x_{2n} \\ \text{subject to} & \underline{\mu}_k \le x_k \le \bar{\mu}_k, \quad k = 1, \ldots, 2n \\ & H(1, x_1, \ldots, x_{2n}) \succeq 0 \end{array}$$

with variables $x_1, \ldots, x_{2n}$.

- consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0 \quad i = 1, \ldots, \ell \\ & h_i(x) = 0 \quad i = 1, \ldots, m \end{array}$$

let p^* denote the optimal value

- define the Lagrangian

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \sum_{i=1}^{\ell} \lambda_i g_i(x) + \sum_{i=1}^{m} \mu_i h_i(x)$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_\ell \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_m \end{bmatrix}$$

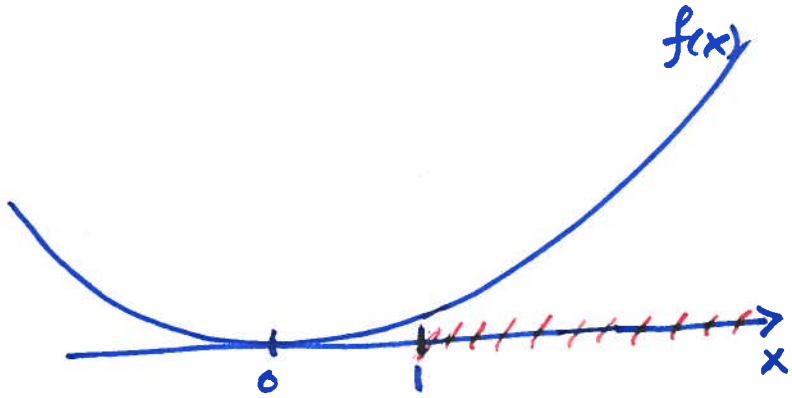
- the (Lagrange) dual function is given by

$$d(\lambda, \mu) = \min_x \mathcal{L}(x, \lambda, \mu)$$

- example:

minimize $f(x) = x^2$

subject to $g(x) = -x + 1 \leq 0$



$$L(x,\lambda) = x^2 + \lambda(-x+1)$$

$$\nabla_x L(x,\lambda) = 2x - \lambda = 0 \rightarrow x = \frac{\lambda}{2}$$

(x where min of tilted f occurs)

$x \geq 1$

$$d(\lambda) = \min_x L(x,\lambda)$$

$$= L\left(\frac{\lambda}{2}, \lambda\right)$$

$$= -\frac{\lambda^2}{4} + \lambda$$

(min value of tilted f)

by inspection

$p^* = 1$

achieved at

$x^* = 1$

(L can be thought of as a "tilted" version of f; d gives the min of L for every amount of tilt)

- suppose $\hat{x}$ is a feasible point for the optimization problem,

  $g_i(\hat{x}) \leq 0$, $h_i(\hat{x}) = 0$, for all i.

  then for any choice of $\lambda \geq 0$ and μ

$$\sum_i \lambda_i g_i(\hat{x}) + \sum_i \mu_i h_i(\hat{x}) \leq 0$$

(≥ 0, ≤ 0, $= 0$)

$$\downarrow$$

$$\mathcal{L}(\hat{x}, \lambda, \mu) = f(\hat{x}) + \sum_i \lambda_i g_i(\hat{x}) + \sum_i \mu_i h_i(\hat{x}) \leq f(\hat{x})$$

(≤ 0)

(tilting reduces value of points inside feasible set)

$$\downarrow$$

$$d(\lambda, \mu) = \min_x \mathcal{L}(x, \lambda, \mu) \leq \mathcal{L}(\hat{x}, \lambda, \mu) \leq f(\hat{x})$$

(it is key that x-hat be feasible and lambda be nonneg.)

since $d(\lambda, \mu) \leq f(\hat{x})$ for every feasible point, we have $d(\lambda, \mu) \leq p^*$.

- the inequality

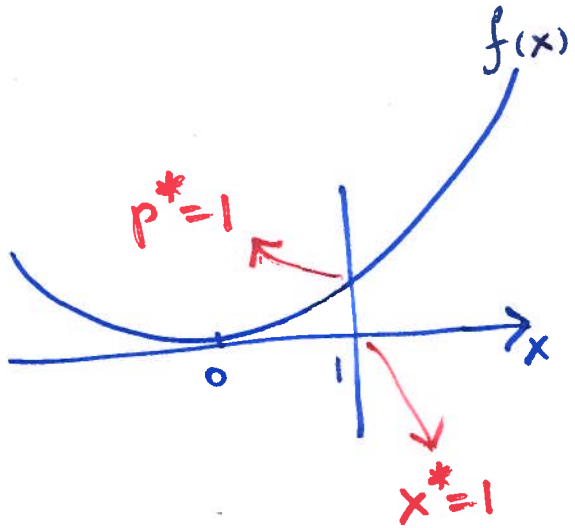
$$d(\lambda, \mu) \leq p^*$$

implies that for any $\lambda \geq 0$ and μ, $d(\lambda, \mu)$ gives a lower bound on the global optimal value.

- example:

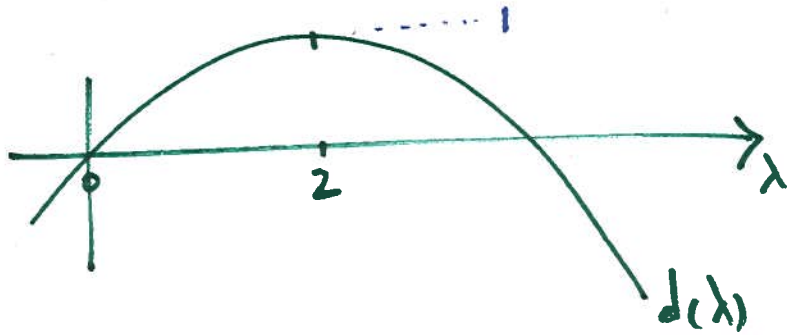
$$f(x) = x^2$$
$$g(x) = -x + 1$$

$$p^* = f(1) = 1$$



$$d(\lambda) = -\frac{\lambda^2}{4} + \lambda \leq 1$$

(in fact, $\max_{\lambda \geq 0} d(\lambda) = 1$)



- the (Lagrange) dual problem to the original problem

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_i(x) = 0 \end{aligned}$$

is

$$\begin{aligned} &\text{maximize} && d(\lambda, \mu) \\ &\text{subject to} && \lambda \geqslant 0 \end{aligned}$$

primal dual

- example:

$$\begin{aligned} &\text{minimize} && x^2 \\ &\text{subject to} && -x + 1 \leq 0 \end{aligned}$$

$$\begin{aligned} &\text{maximize} && d(\lambda) = -\frac{\lambda^2}{4} + \lambda \\ &\text{subject to} && \lambda \geqslant 0 \end{aligned}$$

- one can think of the dual problem as searching for the greatest lower bound to p^*.

- to find the best (largest) lower bound to p^* we solve the Lagrange dual problem

$$\begin{aligned} &\text{maximize} && d(\lambda,\mu) \\ &\text{subject to} && \lambda \geq 0 \end{aligned}$$

- the Lagrange dual problem is a convex optimization problem (concave objective & convex constraints) regardless of whether or not the original (primal) problem is convex: since the dual function is the pointwise infimum of affine functions of (λ,μ),

$$d(\lambda,\mu) = \inf_x L(x,\lambda,\mu) = \inf_x \left(f(x) + \sum_i \lambda_i g_i(x) + \sum_i \mu_i h_i(x)\right)$$

it is concave even when the primal problem is not convex.

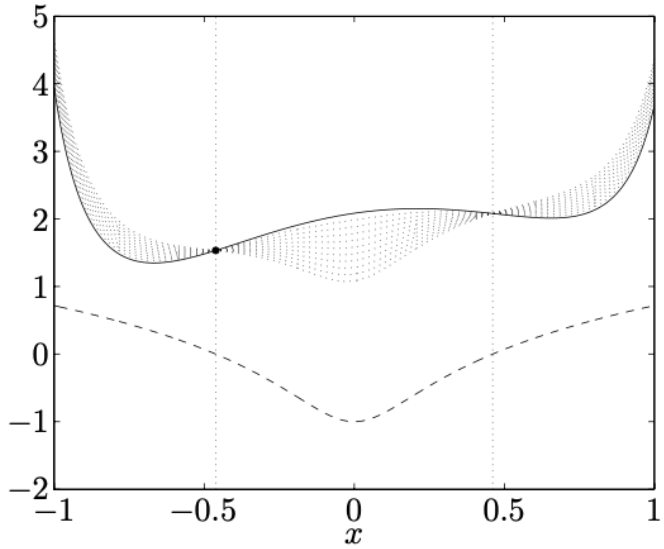


Figure 5.1 *Lower bound from a dual feasible point.* The solid curve shows the objective function f_0, and the dashed curve shows the constraint function f_1. The feasible set is the interval $[-0.46, 0.46]$, which is indicated by the two dotted vertical lines. The optimal point and value are $x^\star = -0.46$, $p^\star = 1.54$ (shown as a circle). The dotted curves show $L(x, \lambda)$ for $\lambda = 0.1,\ 0.2, \ldots, 1.0$. Each of these has a minimum value smaller than $p^\star$, since on the feasible set (and for $\lambda \geq 0$) we have $L(x, \lambda) \leq f_0(x)$.

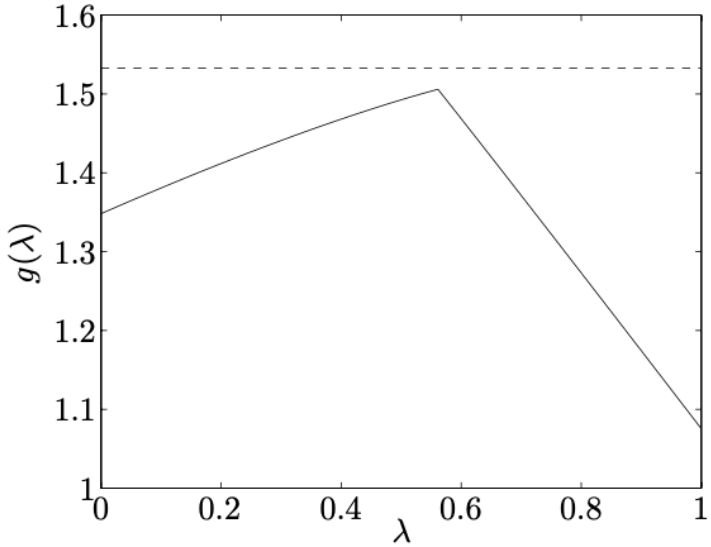


Figure 5.2 The dual function g for the problem in figure 5.1. Neither f_0 nor f_1 is convex, but the dual function is concave. The horizontal dashed line shows $p^\star$, the optimal value of the problem.

• example: we find the Lagrange dual problem for the general

LP problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

$$\mathcal{L}(x,\lambda,\mu) = c^T x + \lambda^T (Gx-h) + \mu^T (Ax-b)$$

aside:

$Ax = b \rightarrow a_i^T x = b_i$

$\mathcal{L} = \ldots + \sum_i \mu_i (a_i^T x - b_i) + \ldots$

$= \ldots + \mu^T (Ax-b) + \ldots$

the dual function then is

$$d(\lambda,\mu) = \inf_x \mathcal{L}(x,\lambda,\mu)$$

$$= \inf_x \left((c + G^T\lambda + A^T\mu)^T x - h^T\lambda - b^T\mu \right)$$

$$d(\lambda,\mu) = \begin{cases} -h^T\lambda - b^T\mu & G^T\lambda + A^T\mu + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

the dual problem is

$$\begin{array}{lll} \text{maximize} & d(\lambda,\mu) = \begin{cases} -h^T\lambda - b^T\mu & G^T\lambda + A^T\mu + c = 0 \\ -\infty & \text{otherwise} \end{cases} \\ \text{subject to} & \lambda \geq 0 \end{array}$$

since d is only finite when $G^T\lambda + A^T\mu + c = 0$ we can form an equivalent problem by making the equality constraints explicit to obtain the LP

$$\begin{array}{ll} \text{maximize} & -h^T\lambda - b^T\mu \\ \text{subject to} & G^T\lambda + A^T\mu + c = 0 \\ & \lambda \geq 0 \end{array}$$

- for simplicity, assume there are no equality constraints.

$$\max_{\lambda \geq 0} \mathcal{L}(x,\lambda) = \max_{\lambda \geq 0} \left(f(x) + \sum_{i=1}^{l} \lambda_i g_i(x) \right)$$

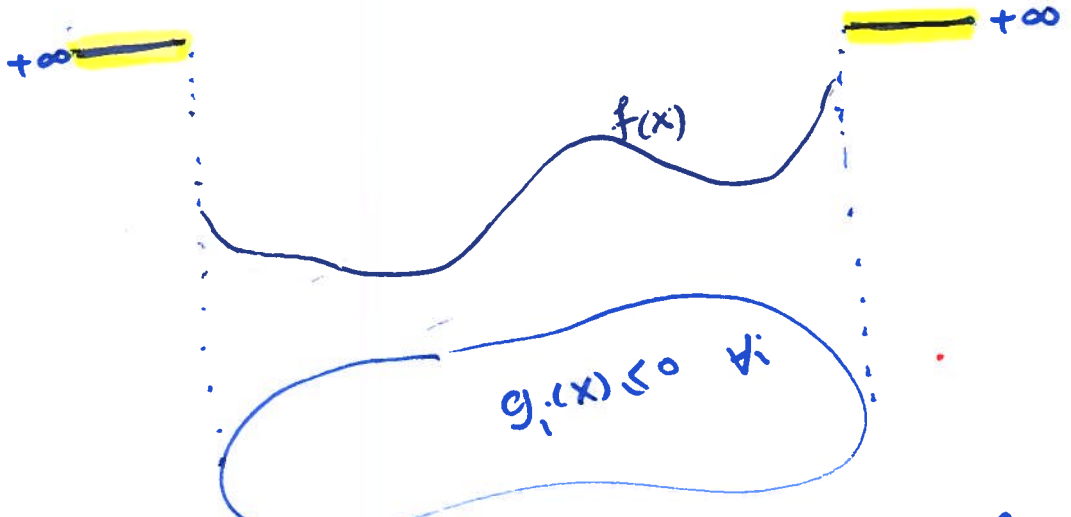
$$= \begin{cases} f(x) & \text{if } g_i(x) \leq 0 \ \forall i \\ \infty & \text{otherwise} \end{cases}$$

(if x not feasible, $g_i(x) > 0$ for some i, then take $\lambda_i \to +\infty$.
if x feasible, $g_i(x) \leq 0$ for all i, then optimal λ is $\lambda = 0$.)

therefore

$$p^* = \min_{x} \max_{\lambda \geq 0} \mathcal{L}(x,\lambda)$$

$f(x)$
s.t. $g_i(x) \leq 0 \ \forall i$



(primal problem can be interpreted as $\mathcal{L}$ having already been maximized in λ.)

- on the other hand, from the definition of dual function

$$\min_x \mathcal{L}(x,\lambda) = d(\lambda)$$

therefore

$$\boxed{q^* = \max_{\lambda \geq 0} \min_x \mathcal{L}(x,\lambda)}$$

(compare with $p^* = \min_x \max_{\lambda \geq 0} \mathcal{L}(x,\lambda)$)

- the min-max inequality says, for any function φ

$$\max_{z \in Z} \min_{w \in W} \varphi(w,z) \leq \min_{w \in W} \max_{z \in Z} \varphi(w,z) \qquad W \subseteq \mathbb{R}^n,\ Z \subseteq \mathbb{R}^n$$

(interpretation: in game, whoever goes second does better)

- application to $\mathcal{L}(x,\lambda)$ implies

$$\boxed{q^* \leq p^*}$$ weak duality!

- weak duality:

$$\max_{\lambda \geq 0} d(\lambda, \mu) = q^* \leq p^*$$

($p^* - q^*$ is called the duality gap.)

- strong duality:

$$q^* = p^*$$

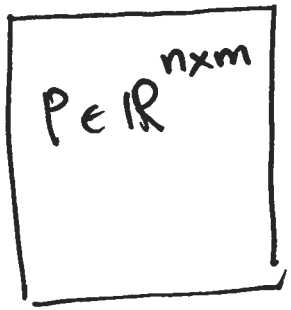
a sufficient condition for strong duality is known as Slater's condition:

if the primal problem is convex and the feasible set contains a strictly feasible point ($g_i(x) \overset{\text{vi}}{<} 0$ not empty) then strong duality holds.

- example: consider a zero-sum matrix game with 2 players.

player 2
chooses column
$\ell \in \{1, \dots, m\}$

player 1
choose row
$k \in \{1, \dots, n\}$



players 1 & 2 have n & m "strategies" respectively.
each player picks his strategy without knowledge of the strategy of his opponent.

if players 1 & 2 pick strategies k & ℓ respectively, then player 1 makes payment $P_{k\ell}$ to player 2.

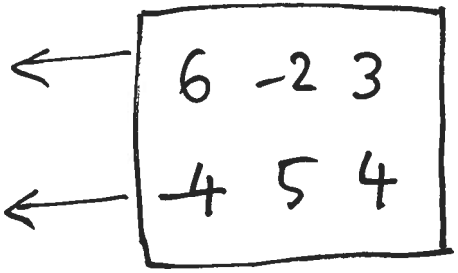
player 1 wants to minimize payment while player 2 wants to maximize it.

① player 1 goes first: player 1 must expect that if he picks row k, player 2 will choose ℓ such that $p_{k\ell}$ is the largest entry in row k; so player 1 must choose that row k which minimizes the row maxima, i.e., find k that solves

$$\underset{k}{\text{minimize}} \; \max_{\ell} p_{k\ell}$$

$k=1 \rightarrow$ row max $= 6$

$k=2 \rightarrow$ row max $= 5$



$k^* = 2 \rightarrow$ min row max $= 5 = p^*$

② player 2 goest first: player 2 expects that if he picks column ℓ, player 1 will choose k so as to minimize $P_{k\ell}$; so player 2 will choose that value of ℓ which maximizes the column minima, i.e., find ℓ that solves

$$\max_{\ell} \min_{k} P_{k\ell}$$

6	-2	3
-4	5	4

$\ell = 1$ → col min $= -4$

$\ell = 2$ → col min $= -2$

$\ell = 3$ → col min $= 3$

$\ell^* = 3$

max col min $= 3 = q^*$

the inequality

$$\max_{\ell}\min_{k} p_{k\ell} \leq \min_{k}\max_{\ell} p_{k\ell}$$

$q^* = 3$ $\qquad$ $p^* = 5$

always holds: its interpretation is that it is always better to go second (to make your choice knowing your opponent's choice)

$k=1 \rightarrow$ row max $= 1$

$k=2 \rightarrow$ row max $= 6$

$k^* = 1$ & $p^* = 1$

-1	1	0
0	4	6

$\ell = 1$ → col min $= -1$

$\ell = 2$ → col min $= 1$

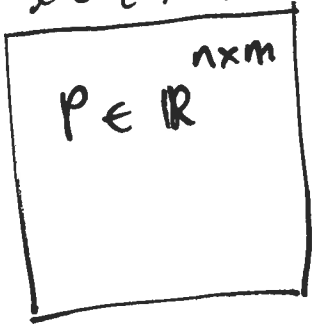
$\ell = 3$ → col min $= 0$

→ $\ell^* = 2$ & $q^* = 1$

• example: consider another zero-sum matrix game with 2 players.

player 1 (player-u) chooses row $k \in \{1, \dots, n\}$

player 2 (player-v) chooses column $l \in \{1, \dots, m\}$



suppose players use randomized ("mixed") strategies (probability distributions for choices)

$$\text{prob}(k=i) = u_i \qquad i=1,\dots,n$$

$$\text{prob}(l=i) = v_i \qquad i=1,\dots,m$$

expected payoff from player-u to player-v is

$$\sum_{k,l} u_k P_{kl} v_l = u^T P v$$

① player-u goes first and chooses u. knowing u, player-v now chooses v to maximize $u^T P v$ for expected payoff

$$\sup \left\{ u^T P v \;\middle|\; v \succcurlyeq 0,\ \mathbb{1}^T v = 1 \right\} = \max_i (P^T u)_i$$

the best thing player-u can do is to choose u to minimize this worst-case payoff to player-v

$$\begin{aligned} \text{minimize} \quad & \max_{i=1,\cdots,m} (P^T u)_i \\ \text{subject to} \quad & u \succcurlyeq 0,\ \mathbb{1}^T u = 1 \end{aligned} \qquad \boxed{1}$$

call the optimal value p_1^*

② player-v goes first and chooses v. knowing v, player-u now chooses u to minimize $u^T P v$ for expected payoff

$$\inf \{ u^T P v \mid u \succeq 0,\ \mathbb{1}^T u = 1 \} = \min_i (P^T v)_i$$

the best thing player-v can do is to choose v to maximize this worst-case payment from player-u

$$\begin{array}{ll} \text{maximize} & \min_{i=1,\dots,n} (Pv)_i \\ \text{subject to} & v \succeq 0,\ \mathbb{1}^T v = 1 \end{array}$$

2

call the optimal value p_2^*.

we investigate the relationship between p_1^* & p_2^* using duality.

problem 1 as LP

rewrite

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && p^T u \preceq t\,\mathbb{1} \\ &&& u \succeq 0,\ \mathbb{1}^T u = 1 \end{aligned}$$

$$\begin{aligned} L &= t + \lambda^T (P^T u - t\,\mathbb{1}) - \eta^T u + \mu(1 - \mathbb{1}^T u) \\ &= \mu + (1 - \mathbb{1}^T\lambda)\,t + (P\lambda - \mu\,\mathbb{1} - \eta)^T u \end{aligned}$$

so the dual function is

$$g(\lambda, \eta, \mu) = \begin{cases} \mu & \mathbb{1}^T\lambda = 1,\ P\lambda - \mu\,\mathbb{1} = \eta \\ -\infty & \text{otherwise} \end{cases}$$

and the dual problem to [1] is

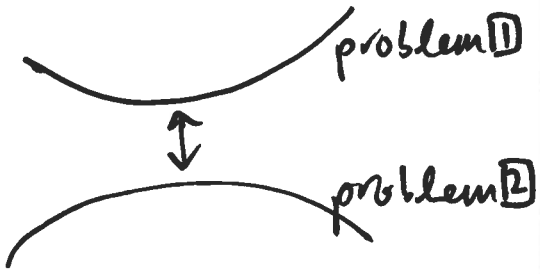
$$\begin{array}{ll} \text{maximize} & \mu \\ \text{subject to} & \lambda \succeq 0,\ \mathbb{1}^T\lambda = 1,\ \eta \succeq 0 \\ & p\lambda - \mu\mathbb{1} = \eta \end{array}$$

eliminating η, we get the equivalent problem

$$\begin{array}{ll} \text{maximize} & \mu \\ \text{subject to} & \lambda \succeq 0,\ \mathbb{1}^T\lambda = 1 \\ & p\lambda \succeq \mu\mathbb{1} \end{array}$$

but this is equivalent to problem [2]. (with $\lambda \sim \nu$)

$p_1^* - p_2^* = \text{gap}$



but since these LPs are feasible, we have strong duality and therefore $\text{gap} = 0 \longrightarrow p_1^* = p_2^*$.

- consider the problem (with differentiable f, g_i, h_i)

minimize $f(x)$
subject to $g_i(x) \leq 0 \quad i=1,\dots,l$
$h_i(x) = 0 \quad i=1,\dots,m$

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = \nabla f(x^*) + \sum_{i=1}^{l} \lambda_i^* \nabla g_i(x^*) + \sum_{i=1}^{m} \mu_i^* \nabla h_i(x^*) = 0$$

$$g_i(x^*) \leq 0 \qquad i=1,\dots,l$$

$$h_i(x^*) = 0 \qquad i=1,\dots,m$$

$$\lambda_i^* \geq 0 \qquad i=1,\dots,l$$

$$\lambda_i^* g_i(x^*) = 0 \qquad i=1,\dots,l$$

(these are known as Karush-Kuhn-Tucker (KKT) conditions)

$$\mathcal{L}(x,\lambda,\mu) = f(x) + \sum_i \lambda_i g_i(x) + \sum_i \mu_i h_i(x).$$

- for general problems the KKT conditions are necessary for x^* to be an optimal point of the primal problem.

- when the primal problem is convex, the KKT conditions are also sufficient for x^* and (λ^*, μ^*) to be primal and dual optimal points with zero duality gap.

recall

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = \nabla_x f(x^*) + \sum_i \lambda_i^* \nabla_x g_i(x^*) + \sum_i \mu_i^* \nabla_x h_i(x^*) = 0$$

$$g_i(x^*) \leq 0 \quad \forall i$$

$$h_i(x^*) = 0 \quad \forall i$$

$$\lambda_i^* \geq 0 \quad \forall i$$

$$\lambda_i^* g_i(x^*) = 0 \quad \forall i$$

(KKT conditions serve as a proxy for solving the dual problem to find the optimal values of lambda and x)

(KKT conditions lend themselves to numerical methods better that the optimality condition Dfo(x)'(y-x)>=0; in fact most algorithms for convex optimization can be interpreted as methods for numerically solving KKT cond.)

- to see how primal/dual optimality and zero duality gap follow from KKT conditions for convex problems, note

  - $\begin{cases} g_i(x^*) \le 0 \quad \forall i \\ h_i(x^*) = 0 \quad \forall i \end{cases} \rightarrow$ x^* is primal feasible

  - $\lambda_i^* \ge 0 \rightarrow \mathcal{L}(x, \lambda^*, \mu^*)$ is convex in x

  - $\nabla_x \mathcal{L}(x, \lambda^*, \mu^*)\Big|_{x=x^*} = 0 \rightarrow$ x^* minimizes $\mathcal{L}(x, \lambda^*, \mu^*)$ over all x.

$$\Downarrow$$

$$d(\lambda^*, \mu^*) = \mathcal{L}(x^*, \lambda^*, \mu^*)$$

$$= f(x^*) + \sum_i \lambda_i^* g_i(x^*) + \sum_i \mu_i^* h_i(x^*)$$

$$= f(x^*)$$

follows from $h_i(x^*) = 0$ & $\lambda_i^* g_i(x^*) = 0$

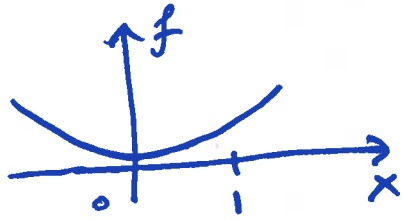
$$\Downarrow$$

x^* & (λ^*, μ^*) have zero duality gap $\rightarrow$ they are primal/dual optimal

aside:
recall that
$d(\lambda, \mu) \le f(x)$
for every $\lambda \ge 0, \mu$
and feasible x.

• example: recall problem

minimize x^2
subject to $x \geq 1$



$p^* = 1$ at $x^* = 1$.

$\mathcal{L}(x,\lambda) = x^2 + \lambda(1-x)$

$\nabla \mathcal{L} = 2x - \lambda = 0 \rightarrow x = \frac{\lambda}{2}$

duality

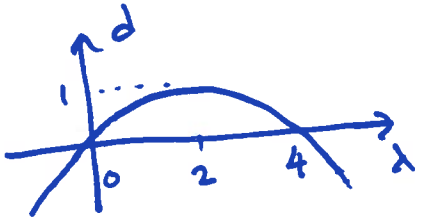
$d(\lambda) = \min_x \mathcal{L}(x,\lambda)$

$= \mathcal{L}(\frac{\lambda}{2},\lambda)$

$= -\frac{\lambda^2}{4} + \lambda$

↓ dual problem

maximize $d(\lambda)$
subject to $\lambda \geq 0$



$q^* = 1$ at $\lambda^* = 2$

KKT

$\nabla \mathcal{L} = 0 \rightarrow x = \frac{\lambda}{2}$

$x \geq 1$

$\lambda \geq 0$

either $x = 1$ or $\lambda = 0$

↙ ↘

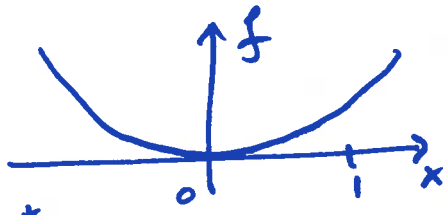
$\frac{\lambda}{2} = 1$ → $\lambda = 2 \geq 0$

$x = \frac{0}{2}$ → $x = 0 \not\geq 1$

$\lambda^* = 2$ optimal amount of "tilt".

- example: consider problem

minimize x^2
subject to $x \leq 1$



$p^* = 0$ at $x^* = 0$

$$\mathcal{L}(x,\lambda) = x^2 + \lambda(x-1)$$

$$\nabla \mathcal{L} = 2x + \lambda = 0 \rightarrow x = -\frac{\lambda}{2}$$

duality

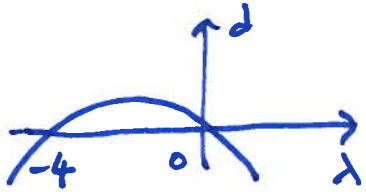
$$d(\lambda) = \min_x \mathcal{L}(x,\lambda)$$

$$= \mathcal{L}\left(-\frac{\lambda}{2}, \lambda\right)$$

$$= -\frac{\lambda^2}{4} - \lambda$$

↓ dual problem

maximize $d(\lambda)$
subject to $\lambda \geq 0$



$q^* = 0$ at $\lambda^* = 0$

KKT

$\nabla \mathcal{L} = 0 \rightarrow x = -\frac{\lambda}{2}$

$x \leq 1$

$\lambda \geq 0$

either $x = 1$ or $\lambda = 0$

$-\frac{\lambda}{2} = 1$

↓

$\lambda = -2 \not\geq 0$

$x = -\frac{0}{2}$

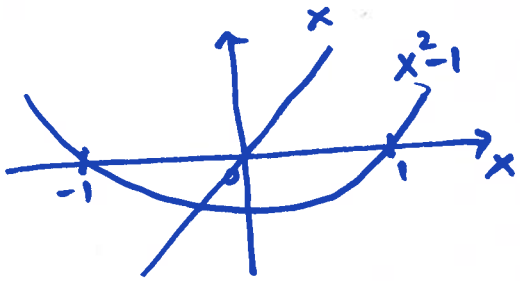
↓

$x = 0 \leq 1$

$\lambda^* = 0$ and no "tilt" necessary
(because constraint inactive at x^*)

• example: consider problem

minimize x
subject to $x^2 \le 1$



$p^* = -1$ at $x^* = -1$

$$\mathcal{L}(x,\lambda) = x + \lambda(x^2-1)$$
$$= \lambda\left[\left(x+\frac{1}{2\lambda}\right)^2 - \frac{1}{4\lambda^2} - 1\right]$$

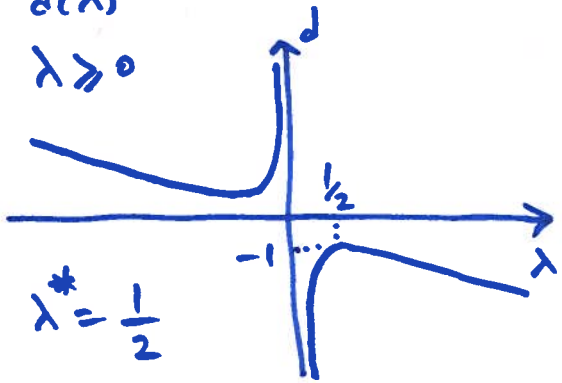
$$\nabla\mathcal{L} = 1 + 2\lambda x = 0 \rightarrow x = -\frac{1}{2\lambda}$$

duality

$$d(\lambda) = \min_x \mathcal{L}(x,\lambda)$$
$$= \mathcal{L}\left(-\frac{1}{2\lambda},\lambda\right)$$
$$= -\frac{1}{4\lambda} - \lambda$$

↓ dual problem

maximize $d(\lambda)$
subject to $\lambda \ge 0$



$q^* = -1$ at $\lambda^* = \frac{1}{2}$

KKT

$$\nabla\mathcal{L} = 0 \rightarrow x = -\frac{1}{2\lambda}$$

$x^2 \le 1$

$\lambda \ge 0$

either $x = \pm 1$ or $\lambda = 0^+$

$x = 1 \rightarrow \lambda = -\frac{1}{2} \not\ge 0$

$x = -1 \rightarrow \lambda = +\frac{1}{2} \ge 0$

$x = -\infty$
not in [-1,+1]

$\lambda^* = \frac{1}{2}$

- example: solve the constrained optimization problem

$$\text{minimize } \|Ax-y\|_2^2$$
$$\text{subject to } \|x\|_2^2 \le 1.$$

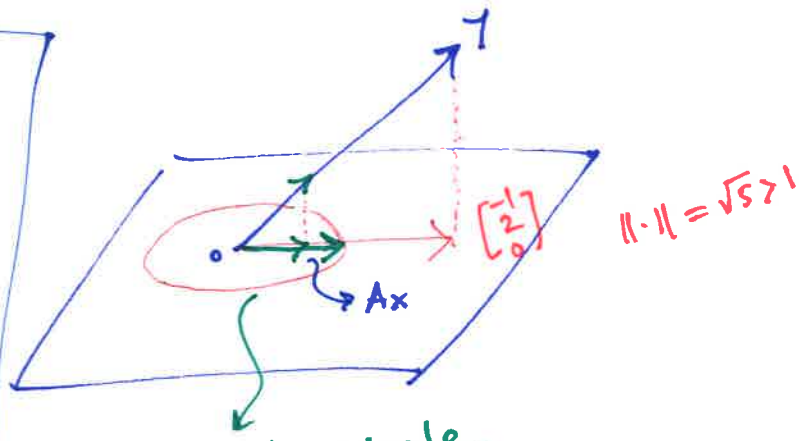
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

aside: if $y = 0.1\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, then x^* would be given by standard least squares, & the constraint would be satisfied with strict inequality.

aside:

$$Ax = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

$$\|x\| \le 1 \iff \|Ax\| \le 1$$



$$\text{minimize } J(x) := \|Ax-y\|_2^2$$
$$\text{subject to } f(x) := \|x\|_2^2 - 1 \le 0$$

$$\mathcal{L} = \|Ax-y\|_2^2 + \lambda(\|x\|_2^2 - 1)$$

KKT conditions:

$$\begin{cases} \nabla\mathcal{L} = 2A^T(Ax-y) + 2\lambda x = 0 \\ \|x\|_2^2 \leq 1 \\ \lambda \geq 0 \\ \lambda f(x) = 0 \rightarrow \begin{cases} \lambda = 0 \\ \text{or} \\ f(x) = 0. \end{cases} \end{cases}$$

$$\nabla\mathcal{L} = 0 \rightarrow (A^TA + \lambda I)x = A^Ty. \rightarrow x = (A^TA + \lambda I)^{-1}A^Ty$$

(notice that if the constraint is inactive and lambda=0 then we recover the standard least-squares solution!)

$$A^+A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow (A^TA + \lambda I)^{-1} = \frac{1}{1+\lambda}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rightarrow (A^TA + \lambda I)^{-1}A^Ty = \frac{1}{1+\lambda}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\|x\|_2^2 \leq 1 \rightarrow \|(A^TA + \lambda I)^{-1}A^Ty\|_2^2 \leq 1$$

20-4

$\|x\|_2^2 \leq 1 \rightarrow \frac{1}{(1+\lambda)^2}(1+4) \leq 1 \rightarrow (1+\lambda)^2 \geq 5.$

$\lambda = 0$ is impossible

$\lambda \neq 0 \rightarrow f(x) = 0 \rightarrow \|x\|_2^2 = 1 \rightarrow \frac{1}{(1+\lambda)^2} 5 = 1 \rightarrow (1+\lambda)^2 = 5$

$\rightarrow 1+\lambda = \pm\sqrt{5} \rightarrow \begin{cases} \lambda = \sqrt{5}-1 > 0 \\ \lambda = -\sqrt{5}-1 < 0 \end{cases}$

$(\lambda > 0)$

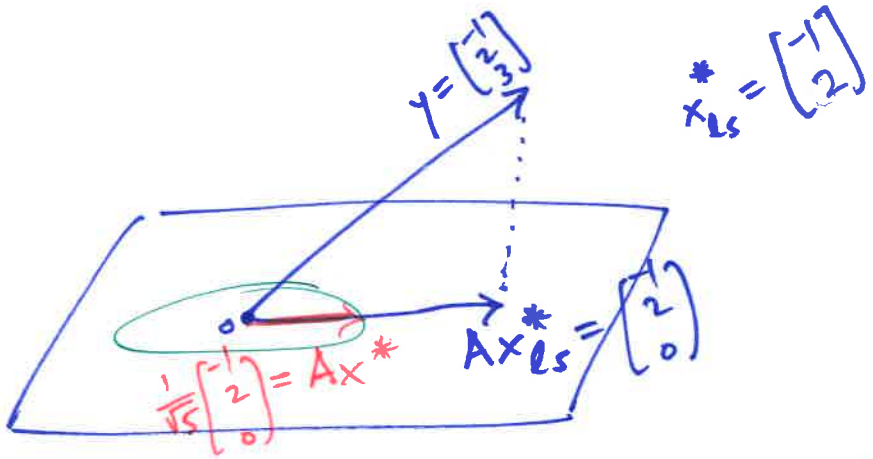
$$x^* = (A^TA + \lambda I)^{-1} A^T y$$

$$= \frac{1}{1+\lambda}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

normalizing factor

(its positivity ensures that x^* is aligned with $x^*_{least\text{-}squares}$)



aside: what would have happened if $y = 0.1\begin{bmatrix}-1\\2\\3\end{bmatrix}$:

$$\|x\|_2^2 \le 1 \rightarrow \frac{1}{(1+\lambda)^2}\, 0.01\,(1+4) \le 1 \rightarrow (1+\lambda)^2 \ge 0.05$$

($\lambda = 0$ is not impossible)

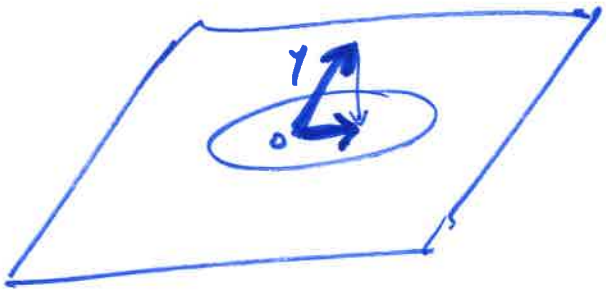
$$\lambda \neq 0 \rightarrow f(x) = 0 \rightarrow \|x\|_2^2 = 1 \rightarrow \frac{0.05}{(1+\lambda)^2} = 1 \rightarrow (1+\lambda)^2 = 0.05$$

$$\rightarrow 1+\lambda = \pm 0.1\sqrt{5}$$ ✗

(λ always negative)

$$\lambda = 0 \rightarrow x^* = (A^TA + \underbrace{\lambda I}_{0})^{-1}A^Ty = (A^TA)^{-1}A^Ty.$$

(just as in case of unconstrained least-squares)



- KKT conditions play an important role in optimization. in a few special cases, it is possible to solve the KKT conditions (and therefore the optimization problem) analytically. but generally these conditions are solved numerically. in fact, most algorithms for convex optimization can be interpreted as methods for numerically solving KKT conditions.

- example: write the KKT conditions for the problem

$$\begin{aligned} &\text{minimize} && \frac{1}{2}x^TPx + q^Tx + r \\ &\text{subject to} && Ax = b \end{aligned}$$

where $P \in S_+^n$.

we have

$$\mathcal{L} = \frac{1}{2} x^T P x + q^T x + r + \mu^T (Ax - b)$$

$$\nabla \mathcal{L} = Px + q + A^T \mu$$

so the KKT conditions are

$$\begin{cases} Px^* + q + A^T \mu^* = 0 \\ Ax^* = b \end{cases}$$

(recall that we derived the same equations earlier, using the nec. & suff. optimality condition $Df_0(x)'(y-x) >= 0$)

or equivalently

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \mu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

- example: write KKT conditions for the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & Gx \preceq h \end{array}$$

$$\mathcal{L} = c^T x + \mu^T (Ax - b) + \lambda^T (Gx - h)$$

$$\nabla \mathcal{L} = c + A^T \mu + G^T \lambda = 0 \quad \longrightarrow \quad A^T \mu + G^T \lambda = -c$$

$$Ax = b$$

$$Gx \preceq h$$

$$\lambda \succeq 0$$

$$\lambda \circ (Gx - h) = 0$$

entrywise product

(also known as Hadamard product)

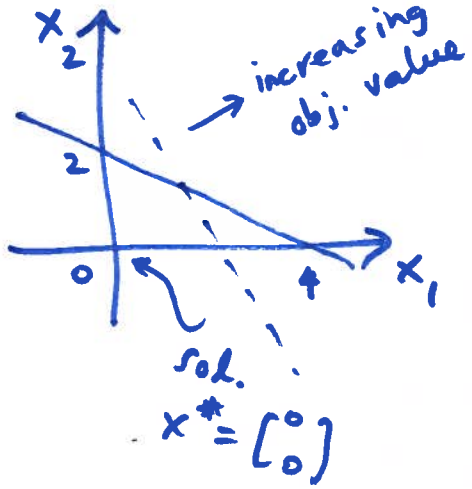
(the last condition is combinatorial in nature; in practice we do not use this approach to solve linear programs)

• example: use KKT conditions to solve the LP

minimize $2x_1 + x_2$

subject to $x_1 \geq 0, \quad x_2 \geq 0$

$x_1 + 2x_2 \leq 4$



$$\mathcal{L}(x, \lambda) = 2x_1 + x_2 + \lambda_1(-x_1) + \lambda_2(-x_2) + \lambda_3(x_1 + 2x_2 - 4)$$

KKT

$$\nabla_x \mathcal{L} = \begin{bmatrix} 2 - \lambda_1 + \lambda_3 \\ 1 - \lambda_2 + 2\lambda_3 \end{bmatrix} = 0 \rightarrow \begin{cases} 2 - \lambda_1 + \lambda_3 = 0 & (1) \\ 1 - \lambda_2 + 2\lambda_3 = 0 & (2) \end{cases}$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$$x_1 \geq 0, x_2 \geq 0, x_1 + 2x_2 \leq 4$$

$$\lambda_1 x_1 = 0 \quad (3) \qquad \lambda_2 x_2 = 0 \quad (4) \qquad \lambda_3(x_1 + 2x_2 - 4) = 0 \quad (5)$$

let's try some points in the feasible set (we can also use knowledge that solution occurs at vertex)

* $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: $x_1 = 1 \xrightarrow{(3)} \lambda_1 = 0 \xrightarrow{(1)} \lambda_3 = -2$

$x_2 = 1 \xrightarrow{(4)} \lambda_2 = 0 \xrightarrow{(2)} \lambda_3 = -1/2$

inconsistent & negative values of λ_3

* $x = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$: $x_1 = 4$, $x_2 = 0$ $\xrightarrow{③}$ $\lambda_1 = 0$ $\xrightarrow{①}$ $\lambda_3 = -2$ $\xrightarrow{②}$ $\lambda_2 = -3$ negative λ_2, λ_3

* $x = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$: $x_1 = 0$, $x_2 = 2$ $\xrightarrow{④}$ $\lambda_2 = 0$ $\xrightarrow{②}$ $\lambda_3 = -\frac{1}{2}$ $\xrightarrow{①}$ $\lambda_1 = \frac{3}{2}$ negative λ_3

* $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: $x_1 = 0$, $x_2 = 0$ $\xrightarrow{⑤}$ $\lambda_3 = 0$ $\xrightarrow{①}$ $\lambda_1 = 2$, $\xrightarrow{②}$ $\lambda_2 = 1$ $\rightarrow$ $\begin{cases} x_1^* = x_2^* = 0 \\ \lambda_1^* = 1, \ \lambda_2^* = 2, \ \lambda_3^* = 0 \end{cases}$

(this is not an efficient method for solving LPs; we'll talk about efficient numerical schemes later)

announcements

- final exam ~~next~~ Thur, Dec 6th, in class
- topics include everything till the end of convex optimization (half way thru lec. 21; till end of chap 4 in book)
- can bring 3 double-sided sheets of notes (total of 6 pages); I will keep these
- exam will have ≈ 4-5 questions
- extra office hours { 2:00-5:00 Wed; 10:30-12:30 Thu
- read hw sol carefully { or email me
- please fill out evaluations online

Unconstrained minimization

$$\text{minimize} \quad f(x)$$

- f convex, twice continuously differentiable
- we assume optimal value $p^\star = \inf_x f(x)$ is attained (and finite)

unconstrained minimization methods

- produce sequence of points $x^{(k)} \in \mathbf{dom}\, f$, $k = 0, 1, \ldots$ with

$$f(x^{(k)}) \to p^\star$$

- can be interpreted as iterative methods for solving optimality condition

$$\nabla f(x^\star) = 0$$

Descent methods

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)} \quad \text{with } f(x^{(k+1)}) < f(x^{(k)})$$

- other notations: $x^+ = x + t\Delta x$, $x := x + t\Delta x$
- Δx is the *step*, or *search direction*; t is the *step size*, or *step length*
- from convexity, $f(x^+) < f(x)$ implies $\nabla f(x)^T \Delta x < 0$ (*i.e.*, Δx is a *descent direction*)

General descent method.

given a starting point $x \in \mathbf{dom}\, f$.
repeat

1. Determine a descent direction Δx.
2. *Line search.* Choose a step size $t > 0$.
3. *Update.* $x := x + t\Delta x$.

until stopping criterion is satisfied.

Line search types

exact line search: $t = \operatorname{argmin}_{t>0} f(x + t\Delta x)$

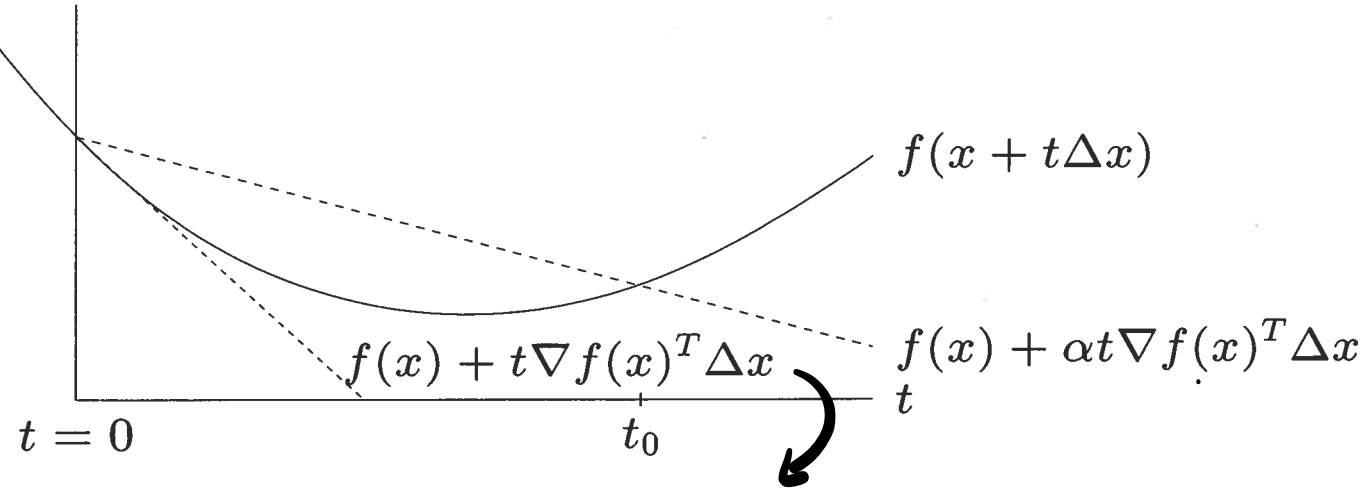
backtracking line search (with parameters $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$)

- starting at $t = 1$, repeat $t := \beta t$ until

$$f(x + t\Delta x) < f(x) + \underbrace{\alpha t \nabla f(x)^T \Delta x}_{<0}$$

(guarantees that f(x+tdx)<f(x) because dx is descent direction and thus Df(x)'dx<0)

- graphical interpretation: backtrack until $t \leq t_0$



(follows from first-order condition for convexity with y=x+tdx)

Gradient descent method

general descent method with $\Delta x = -\nabla f(x)$

---

given a starting point $x \in \mathbf{dom}\, f$.
repeat

1. $\Delta x := -\nabla f(x)$.
2. *Line search.* Choose step size t via exact or backtracking line search.
3. *Update.* $x := x + t\Delta x$.

until stopping criterion is satisfied.

- stopping criterion usually of the form $\|\nabla f(x)\|_2 \leq \epsilon$
- convergence result: for strongly convex f,

$$f(x^{(k)}) - p^\star \leq c^k (f(x^{(0)}) - p^\star)$$

(strongly convex $\nabla^2 f(x) \succeq mI$)

$c \in (0,1)$ depends on m, $x^{(0)}$, line search type

- very simple, but often very slow; rarely used in practice

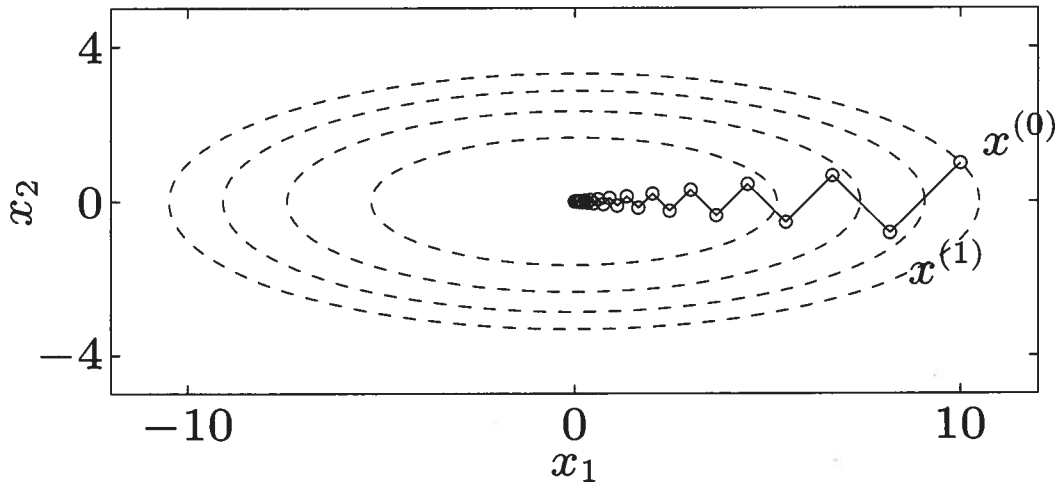
quadratic problem in $\mathbf{R}^2$

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1} \right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1} \right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:



Newton step

$$\Delta x_{\rm nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

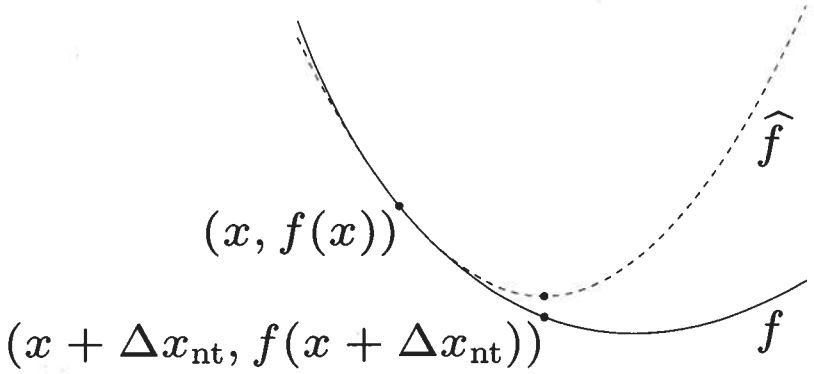
interpretations

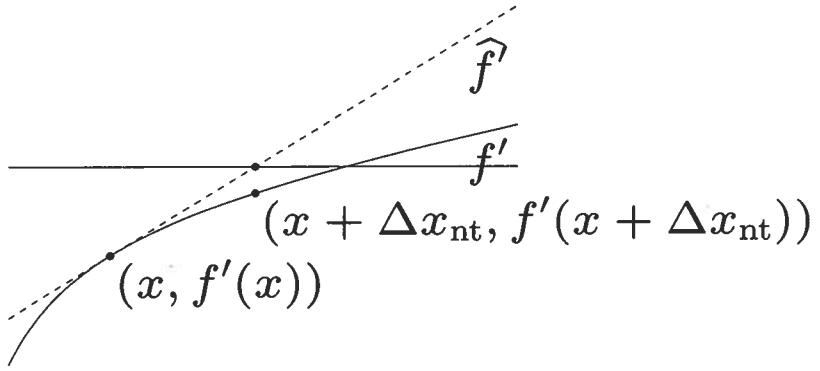
- $x + \Delta x_{\rm nt}$ minimizes second order approximation

$$\widehat{f}(x+v) = f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v$$

- $x + \Delta x_{\rm nt}$ solves linearized optimality condition

$$\nabla f(x+v) \approx \nabla \widehat{f}(x+v) = \nabla f(x) + \nabla^2 f(x) v = 0$$





Newton decrement

$$\lambda(x) = \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)\right)^{1/2}$$

a measure of the proximity of x to $x^\star$

properties

- gives an estimate of $f(x) - p^\star$, using quadratic approximation $\widehat{f}$:

$$f(x) - \inf_y \widehat{f}(y) = \frac{1}{2}\lambda(x)^2$$

- equal to the norm of the Newton step in the quadratic Hessian norm

$$\lambda(x) = \left(\Delta x_{\rm nt}^T \nabla^2 f(x) \Delta x_{\rm nt}\right)^{1/2}$$

- directional derivative in the Newton direction: $\nabla f(x)^T \Delta x_{\rm nt} = -\lambda(x)^2$

Newton's method

---

given a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.
repeat

1. *Compute the Newton step and decrement.*
   $$\Delta x_{\rm nt} := -\nabla^2 f(x)^{-1} \nabla f(x); \quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x).$$
2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\rm nt}$.

Classical convergence analysis

assumptions

- f strongly convex on S with constant m: $\nabla^2 f(x) \succeq mI \quad \forall x \in S$
- $\nabla^2 f$ is Lipschitz continuous on S, with constant $L > 0$:

$$\|\nabla^2 f(x) - \nabla^2 f(y)\|_2 \leq L\|x - y\|_2$$

(L measures how well f can be approximated by a quadratic function)

outline: there exist constants $\eta \in (0, m^2/L)$, $\gamma > 0$ such that

- if $\|\nabla f(x)\|_2 \geq \eta$, then $f(x^{(k+1)}) - f(x^{(k)}) \leq -\gamma$
- if $\|\nabla f(x)\|_2 < \eta$, then

$$\frac{L}{2m^2}\|\nabla f(x^{(k+1)})\|_2 \leq \left(\frac{L}{2m^2}\|\nabla f(x^{(k)})\|_2\right)^2$$

$< \frac{1}{2}$

damped Newton phase ($\|\nabla f(x)\|_2 \geq \eta$)

- most iterations require backtracking steps
- function value decreases by at least γ
- if $p^\star > -\infty$, this phase ends after at most $(f(x^{(0)}) - p^\star)/\gamma$ iterations

quadratically convergent phase ($\|\nabla f(x)\|_2 < \eta < m^2/L$)

- all iterations use step size $t = 1$
- $\|\nabla f(x)\|_2$ converges to zero quadratically: if $\|\nabla f(x^{(k)})\|_2 < \eta$, then

$$\frac{L}{2m^2}\|\nabla f(x^l)\|_2 \leq \left(\frac{L}{2m^2}\|\nabla f(x^k)\|_2\right)^{2^{l-k}} \leq \left(\frac{1}{2}\right)^{2^{l-k}}, \qquad l \geq k$$

conclusion: number of iterations until $f(x) - p^\star \leq \epsilon$ is bounded above by

$$\frac{f(x^{(0)}) - p^\star}{\gamma} + \log_2 \log_2(\epsilon_0/\epsilon)$$

- γ, ϵ_0 are constants that depend on m, L, $x^{(0)}$
- second term is small (of the order of 6) and almost constant for practical purposes
- in practice, constants m, L (hence γ, ϵ_0) are usually unknown
- provides qualitative insight in convergence properties ($i.e.$, explains two algorithm phases)

Equality constrained minimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- f convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- we assume $p^\star$ is finite and attained

optimality conditions: $x^\star$ is optimal iff there exists a $\nu^\star$ such that

$$\nabla f(x^\star) + A^T \nu^\star = 0, \qquad Ax^\star = b$$

(recall that we derived these equations using the nec. & suff. optimality condition Df(x)'(y-x)>=0; one can also derive them using KKT conditions with L=f+v'(Ax-b) and hence DL=Df+A'v=0)

equality constrained quadratic minimization (with $P \in \mathbf{S}_+^n$)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

optimality condition:

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ \nu^\star \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

(recall from KKT/duality lecture)

- coefficient matrix is called KKT matrix
- KKT matrix is nonsingular if and only if

$$Ax = 0, \quad x \neq 0 \quad \Longrightarrow \quad x^T Px > 0$$

- equivalent condition for nonsingularity: $P + A^T A \succ 0$

Eliminating equality constraints

represent solution of $\{x \mid Ax = b\}$ as

$$\{x \mid Ax = b\} = \{Fz + \hat{x} \mid z \in \mathbf{R}^{n-p}\}$$

- $\hat{x}$ is (any) particular solution
- range of $F \in \mathbf{R}^{n \times (n-p)}$ is nullspace of A ($\mathbf{rank}\, F = n - p$ and $AF = 0$)

reduced or eliminated problem

$$\text{minimize} \quad f(Fz + \hat{x})$$

- an unconstrained problem with variable $z \in \mathbf{R}^{n-p}$
- from solution $z^\star$, obtain $x^\star$ and $\nu^\star$ as

$$x^\star = Fz^\star + \hat{x}, \qquad \nu^\star = -(AA^T)^{-1}A\nabla f(x^\star)$$

example: optimal allocation with resource constraint

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) \\ \text{subject to} & x_1 + x_2 + \cdots + x_n = b \end{array}$$

eliminate $x_n = b - x_1 - \cdots - x_{n-1}$, *i.e.*, choose

$$\hat{x} = be_n, \qquad F = \begin{bmatrix} I \\ -\mathbf{1}^T \end{bmatrix} \in \mathbf{R}^{n \times (n-1)}$$

reduced problem:

$$\text{minimize} \quad f_1(x_1) + \cdots + f_{n-1}(x_{n-1}) + f_n(b - x_1 - \cdots - x_{n-1})$$

(variables $x_1, \ldots, x_{n-1}$)

Newton step

Newton step $\Delta x_{\rm nt}$ of f at feasible x is given by solution v of

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

interpretations

- $\Delta x_{\rm nt}$ solves second order approximation (with variable v, $x^* \approx x + v$)

$$\begin{array}{ll} \text{minimize} & \widehat{f}(x+v) = f(x) + \nabla f(x)^T v + (1/2) v^T \nabla^2 f(x) v \\ \text{subject to} & A(x+v) = b \end{array}$$

- $\Delta x_{\rm nt}$ equations follow from linearizing optimality conditions $\left(x^* \approx x + v \right)$

$$\nabla f(x+v) + A^T w \approx \nabla f(x) + \nabla^2 f(x) v + A^T w = 0, \qquad A(x+v) = b$$

Newton decrement

$$\lambda(x) = \left(\Delta x_{\rm nt}^T \nabla^2 f(x) \Delta x_{\rm nt}\right)^{1/2} = \left(-\nabla f(x)^T \Delta x_{\rm nt}\right)^{1/2}$$

properties

- gives an estimate of $f(x) - p^\star$ using quadratic approximation $\widehat{f}$:

$$f(x) - \inf_{Ay=b} \widehat{f}(y) = \frac{1}{2}\lambda(x)^2$$

- directional derivative in Newton direction:

$$\frac{d}{dt} f(x + t\Delta x_{\rm nt})\Big|_{t=0} = -\lambda(x)^2$$

- in general, $\lambda(x) \neq \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)\right)^{1/2}$

Newton's method with equality constraints

---

given starting point $x \in \mathbf{dom}\, f$ with $Ax = b$, tolerance $\epsilon > 0$.

repeat

1. Compute the Newton step and decrement Δx_{nt}, $\lambda(x)$.
2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$.

---

a feasible descent method: $x^{(k)}$ feasible and $f(x^{(k+1)}) < f(x^{(k)})$

Newton step at infeasible points

2nd interpretation of page 11–6 extends to infeasible x (*i.e.*, $Ax \neq b$)

linearizing optimality conditions at infeasible x (with $x \in \mathbf{dom}\, f$) gives

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax - b \end{bmatrix} \tag{1}$$

primal-dual interpretation

- write optimality condition as $r(y) = 0$, where

$$y = (x, \nu), \qquad r(y) = (\nabla f(x) + A^T \nu, Ax - b)$$

- linearizing $r(y) = 0$ gives $r(y + \Delta y) \approx r(y) + Dr(y)\Delta y = 0$:

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ \Delta \nu_{\rm nt} \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + A^T \nu \\ Ax - b \end{bmatrix}$$

  same as (1) with $w = \nu + \Delta \nu_{\rm nt}$

Infeasible start Newton method

---

given starting point $x \in \mathbf{dom}\, f$, ν, tolerance $\epsilon > 0$, $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$.

repeat

1. Compute primal and dual Newton steps $\Delta x_{\rm nt}$, $\Delta \nu_{\rm nt}$.
2. *Backtracking line search on* $\|r\|_2$.

   $t := 1$.

   while $\|r(x + t\Delta x_{\rm nt}, \nu + t\Delta \nu_{\rm nt})\|_2 > (1 - \alpha t)\|r(x, \nu)\|_2$, $t := \beta t$.
3. *Update.* $x := x + t\Delta x_{\rm nt}$, $\nu := \nu + t\Delta \nu_{\rm nt}$.

until $Ax = b$ and $\|r(x, \nu)\|_2 \leq \epsilon$.

- not a descent method: $f(x^{(k+1)}) > f(x^{(k)})$ is possible
- directional derivative of $\|r(y)\|_2$ in direction $\Delta y = (\Delta x_{\rm nt}, \Delta \nu_{\rm nt})$ is

$$\frac{d}{dt}\|r(y + t\Delta y)\|_2\Big|_{t=0} = -\|r(y)\|_2$$

Inequality constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{1}$$

- f_i convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- we assume $p^\star$ is finite and attained
- we assume problem is strictly feasible: there exists $\tilde{x}$ with

$$\tilde{x} \in \mathbf{dom}\, f_0, \qquad f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \qquad A\tilde{x} = b$$

hence, strong duality holds and dual optimum is attained

Logarithmic barrier

reformulation of (1) via indicator function:

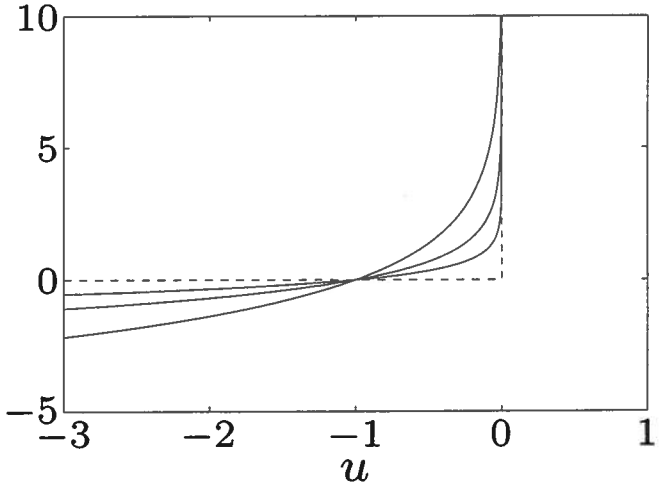
$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

where $I_-(u) = 0$ if $u \leq 0$, $I_-(u) = \infty$ otherwise (indicator function of $\mathbf{R}_-$)

approximation via logarithmic barrier

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t) \sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of I_-
- approximation improves as $t \to \infty$



- numerical methods for solving LPs.
  - simplex method.
  - interior point method

minimize $c^T x$
subject to $Ax \leq b$.

$$\Rightarrow \begin{cases} a_1^T x \leq b_1 \\ \vdots \\ a_m^T x \leq b_m \end{cases}$$

$$\Downarrow$$

$$\text{minimize} \quad c^T x - \frac{1}{t} \sum_{i=1}^{m} \underbrace{\log(b_i - a_i^T x)}_{\text{"log barrier"}}$$

$t \to \infty$.

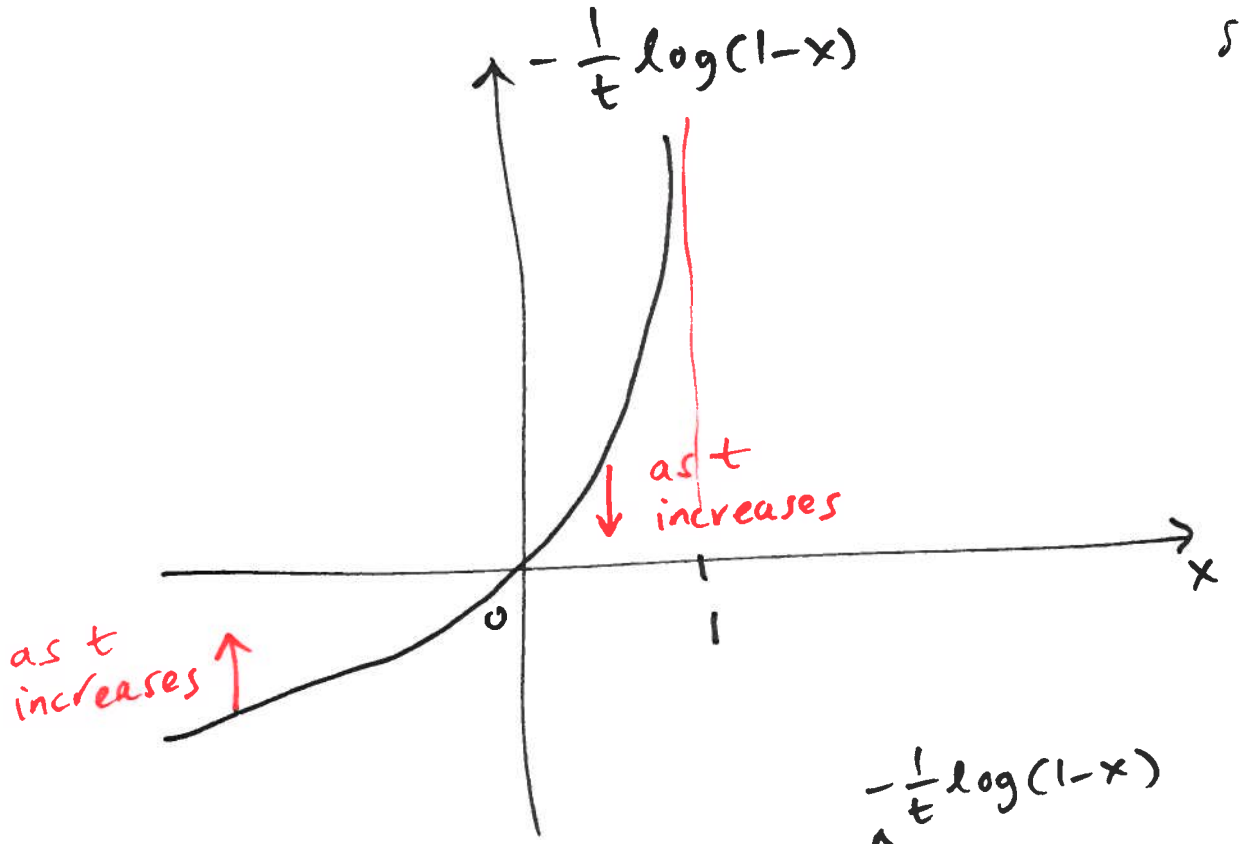
suppose $x \in \mathbb{R}$, $a=1$, $b=1$

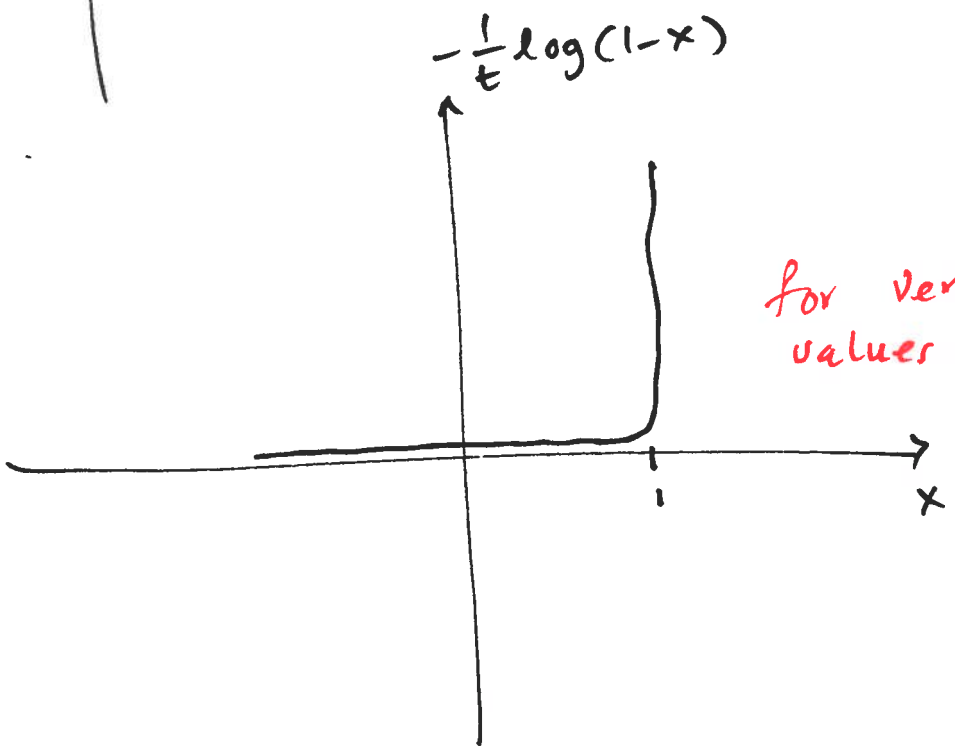
$\downarrow$

$x \leq 1$

$\downarrow$

$x - 1 \leq 0$





for very large values of t.

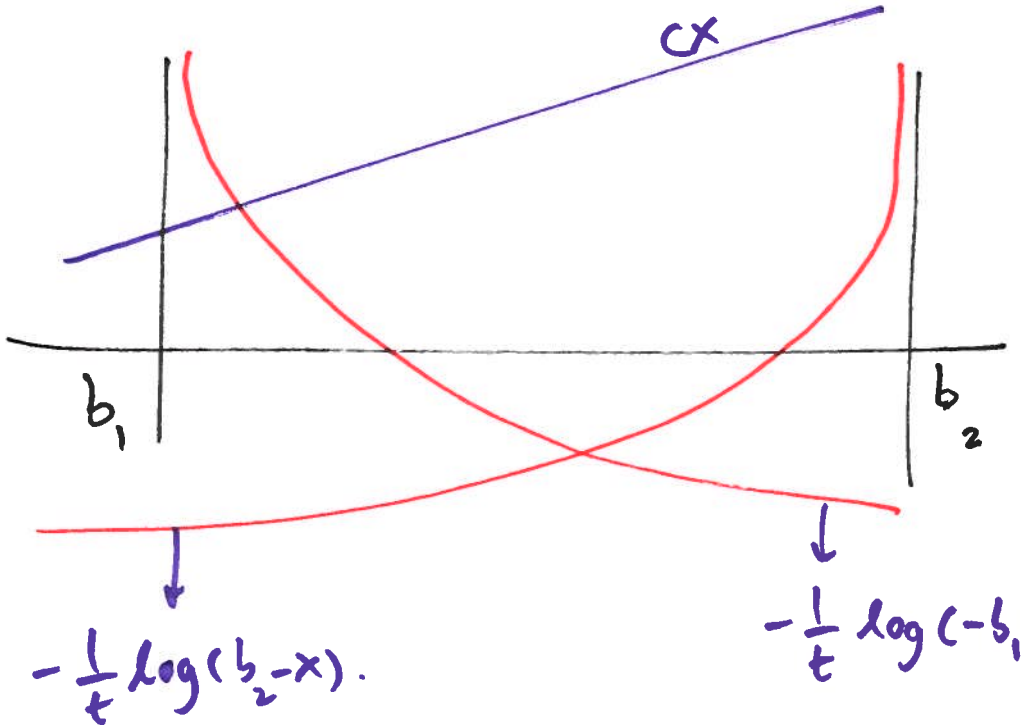
for example, suppose we want to solve

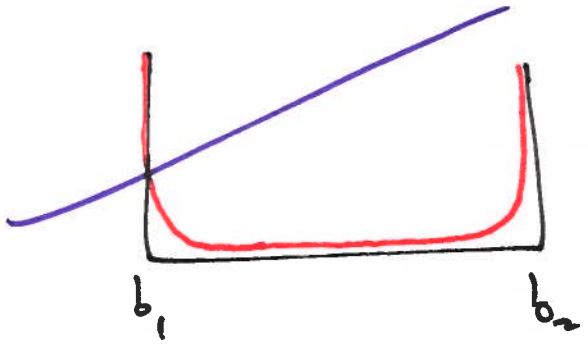
minimize cx

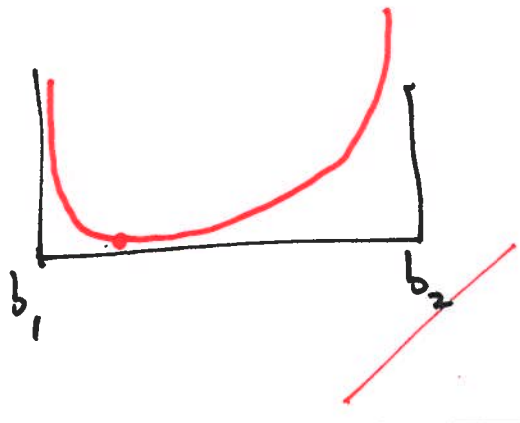
subject to $b_1 \leq x \leq b_2$

$\downarrow$

minimize $cx - \frac{1}{t}\log(-b_1+x) - \frac{1}{t}\log(b_2-x)$.







logarithmic barrier function

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)), \quad \mathbf{dom}\, \phi = \{x \mid f_1(x) < 0, \ldots, f_m(x) < 0\}$$

- convex (follows from composition rules)
- twice continuously differentiable, with derivatives

$$\begin{aligned}
\nabla\phi(x) &= \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \\
\nabla^2\phi(x) &= \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)
\end{aligned}$$

Barrier method

---

given strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$.

repeat

1. *Centering step.* Compute $x^\star(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. *Update.* $x := x^\star(t)$.
3. *Stopping criterion.* **quit** if $m/t < \epsilon$.
4. *Increase* t. $t := \mu t$.

- terminates with $f_0(x) - p^\star \leq \epsilon$ (stopping criterion follows from $f_0(x^\star(t)) - p^\star \leq m/t$)
- centering usually done using Newton's method, starting at current x
- choice of μ involves a trade-off: large μ means fewer outer iterations, more inner (Newton) iterations; typical values: $\mu = 10$–20
- several heuristics for choice of $t^{(0)}$

- interior point algorithm to solve LPs

1. begin with some guess in the feasible set.

2. using variational methods, find the gradient & Hessian of the objective

$$J(x) = c^T x - \frac{1}{t} \sum_{i=1}^{m} \log(b_i - a_i^T x)$$

$$\nabla J = c + \frac{1}{t} \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x} a_i$$

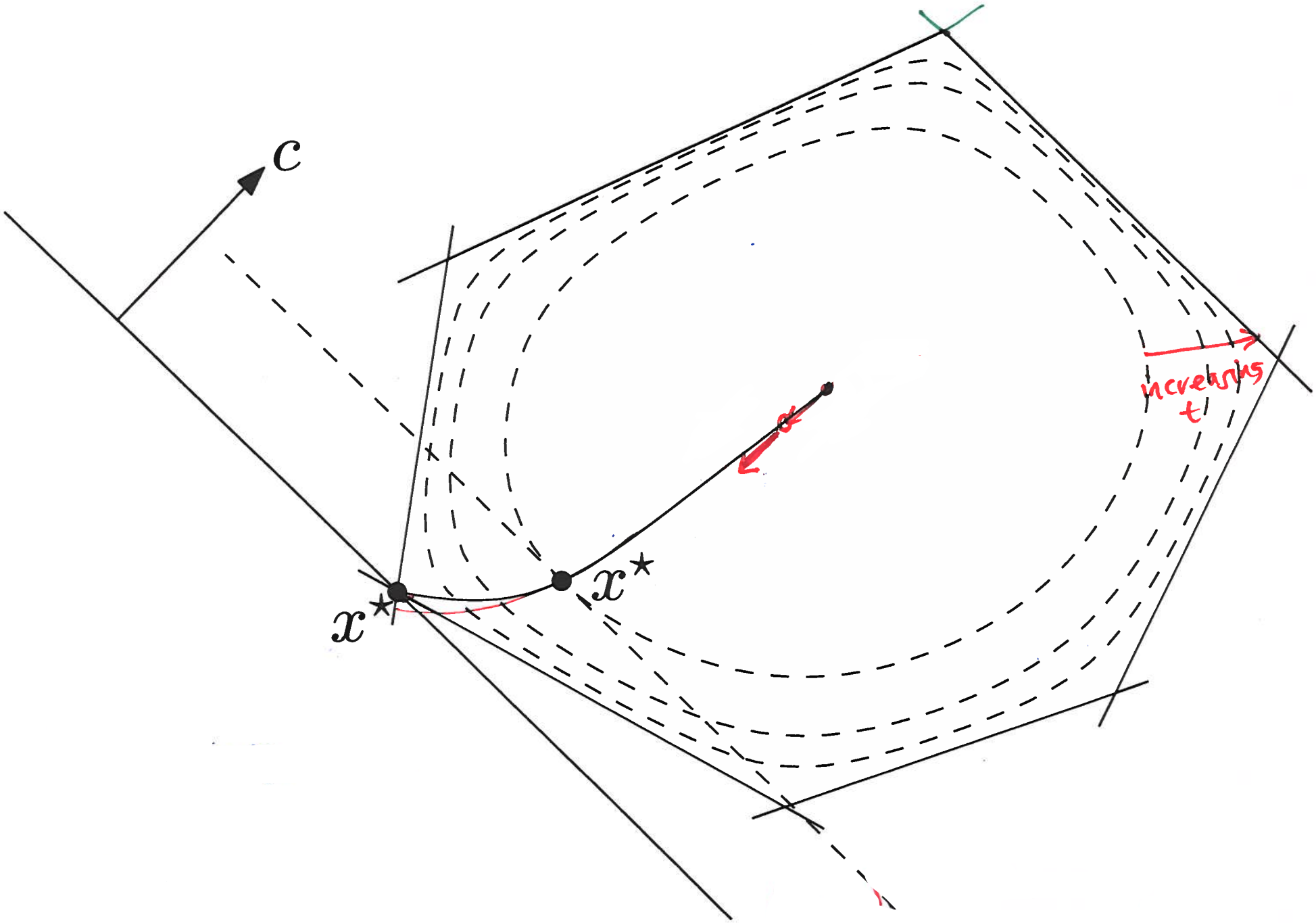
$$\nabla^2 J = \frac{1}{t} \sum_{i=1}^{m} \left(\frac{1}{b_i - a_i^T x} \right)^2 a_i a_i^T$$

use $\log(1-x) \approx -x - \frac{1}{2}x^2 + \text{h.o.t.}$

to update our estimate of the optimal x

3. increase t.

4. go back to step 2 & repeat untill stopping criterion satisfied.


c
$x^\star$
$x^\star$
increasing
t