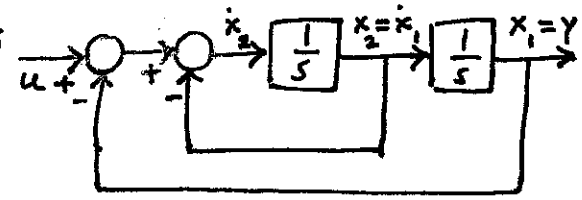
Solutions to Homework 9

①

We redraw the block diagram as follows



From the diagram we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

The observer equations are

$$\dot{\hat{x}}=(A-LC)\hat{x}+Bu+Ly \qquad L=\begin{bmatrix}\ell_1\\ \ell_2\end{bmatrix}.$$

$$A-LC=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}-\begin{bmatrix}\ell_1\\ \ell_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}=\begin{bmatrix}-\ell_1 & 1\\ -1-\ell_2 & -1\end{bmatrix}.$$

$$\dot{\hat{x}}=\begin{bmatrix}-\ell_1 & 1\\ -1-\ell_2 & -1\end{bmatrix}\hat{x}+\begin{bmatrix}0\\ 1\end{bmatrix}u+\begin{bmatrix}\ell_1\\ \ell_2\end{bmatrix}y.$$

$$\det\left[sI-(A-LC)\right]=\det\begin{bmatrix}s+\ell_1 & -1\\ 1+\ell_2 & s+1\end{bmatrix}=(s+\ell_1)(s+1)+1+\ell_2$$

$$=s^2+(\ell_1+1)s+1+\ell_1+\ell_2=0$$

The desired characteristic equation is

$$(s+5)^2=s^2+10s+25=0$$

$$\Rightarrow\begin{cases}\ell_1+1=10\\ 1+\ell_1+\ell_2=25\end{cases}\Rightarrow\begin{cases}\ell_1=9\\ \ell_2=15\end{cases}$$

②

The ship dynamics are

$$\begin{bmatrix} \dot{d} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d \\ v \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} d \\ v \end{bmatrix}.$$

Notice that the input $u(t)$ is identically zero.

The observer can be written as

$$\begin{bmatrix} \dot{\hat{d}} \\ \dot{\hat{v}} \end{bmatrix} = \left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{d} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} y$$

$$= \begin{bmatrix} -\ell_1 & -1 \\ -\ell_2 & 0 \end{bmatrix} \begin{bmatrix} \hat{d} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} y.$$

The characteristic equation of the error dynamics is

$$\det\left[sI - (A - LC) \right] = \det \begin{bmatrix} s+\ell_1 & 1 \\ \ell_2 & s \end{bmatrix} = s(s+\ell_1) - \ell_2$$

$$= s^2 + \ell_1 s - \ell_2 = 0$$

Comparing to the desired characteristic equation, we get

$$\begin{cases} \ell_1 = 10 \\ -\ell_2 = 25 \end{cases} \Rightarrow \begin{cases} \ell_1 = 10 \\ \ell_2 = -25 \end{cases}.$$

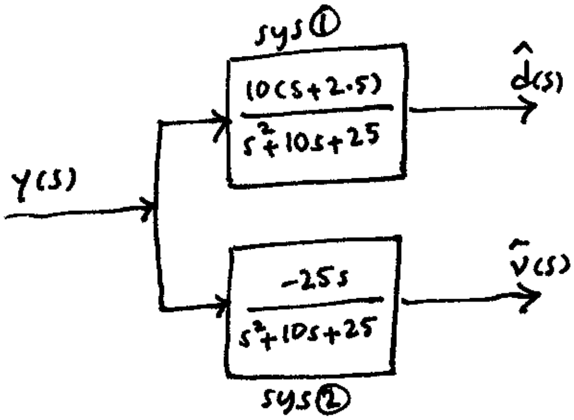
Therefore the observer dynamics are

$$\begin{bmatrix} \dot{\hat{d}} \\ \dot{\hat{v}} \end{bmatrix} = \begin{bmatrix} -10 & -1 \\ 25 & 0 \end{bmatrix} \begin{bmatrix} \hat{d} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} 10 \\ -25 \end{bmatrix} y.$$

The transfer function from $y(s)$ to $\begin{bmatrix} \hat{d}(s) \\ \hat{v}(s) \end{bmatrix}$ is

$$\begin{bmatrix} \hat{d}(s) \\ \hat{v}(s) \end{bmatrix} = \begin{bmatrix} s+10 & 1 \\ -25 & s \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ -25 \end{bmatrix} y(s) = \frac{1}{s(s+10)+25} \begin{bmatrix} s & -1 \\ 25 & s+10 \end{bmatrix} \begin{bmatrix} 10 \\ -25 \end{bmatrix} y(s)$$

$$= \begin{bmatrix} \dfrac{10(s+2.5)}{s^2+10s+25} \\[2ex] \dfrac{-25s}{s^2+10s+25} \end{bmatrix} y(s)$$

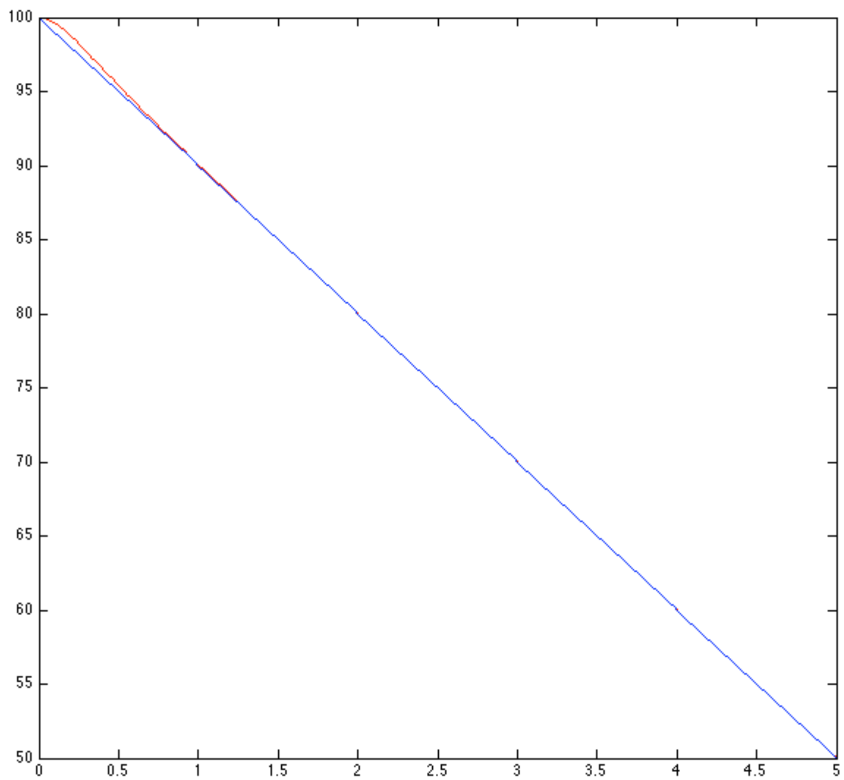


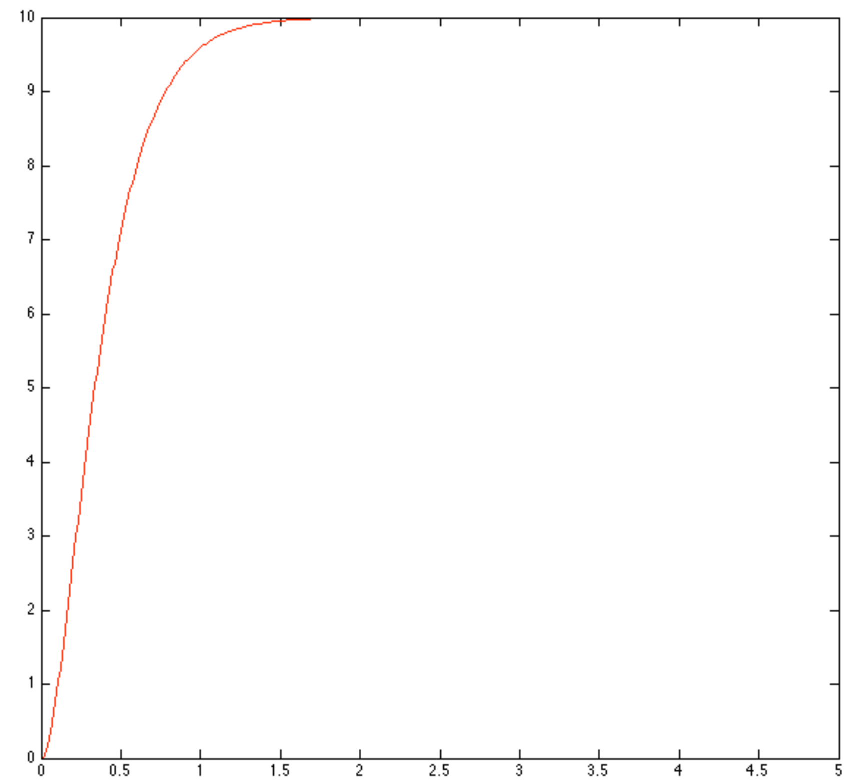
Note that system ① is basically a low-pass filter with unit DC-gain (i.e., $\left.\dfrac{10(s+2.5)}{s^2+10s+25}\right|_{s=0} = 1$). And system ② is a low-pass-filtered differentiator (the "s" in the numerator is a differentiator; this is expected, since the input to system ② is $d(t)$ and the output is $\hat{v}(t)$).

The following code simulates the observer whose initial conditions have been set to

$$\begin{bmatrix} \hat{d}(0) \\ \hat{v}(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

```
t = 0:0.01:5;

y = 100 - 10*t;
xhat0 = [100;0];

A = [-10, -1; 25, 0];
L = [10; -25];
C = eye(2);

yhat = lsim(A,L,C,0,y,t,xhat0);

figure
subplot(1,2,1)
plot(t,yhat(:,1),'r',t,y,'b')
subplot(1,2,2)
plot(t,yhat(:,2),'r',t,10,'b')
```


100
95
90
85
80
75
70
65
60
55
50
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


10
9
8
7
6
5
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

On the left plot the blue line represents $y(t) = d(t)$, the actual location of the ship. The red line is the estimate $\hat{d}(t)$. It is clear that the estimate $\hat{d}(t)$ converges to the actual distance $d(t)$ very fast. The plot on the right shows $\hat{v}(t)$. Note that the actual velocity of the ship is 10. Again, the estimator is converging to the true velocity in about 1.5 sec.

Note that the reason for the small discrepancy between $\hat{d}$ and d is the fact that the observer was initialized at the correct value of $\hat{d}(0) = 100 = d(0)$. And the large (initial) discrepancy between $\hat{v}$ and v is due to the large difference between $\hat{v}(0) = 0$ and $v(0) = 10$.

Finally, recall that $\begin{bmatrix} \dot{\hat{d}}(t) \\ \dot{\hat{v}}(t) \end{bmatrix} = \begin{bmatrix} -10 & -1 \\ 25 & 0 \end{bmatrix} \begin{bmatrix} \hat{d}(t) \\ \hat{v}(t) \end{bmatrix} + \begin{bmatrix} 10 \\ -25 \end{bmatrix} (100 - 10t)$,

and therefore setting $t = 0$ we get

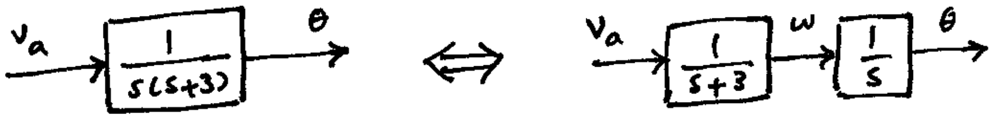
$$\begin{bmatrix} \dot{\hat{d}}(0) \\ \dot{\hat{v}}(0) \end{bmatrix} = \begin{bmatrix} -10 & -1 \\ 25 & 0 \end{bmatrix} \begin{bmatrix} \hat{d}(0) \\ \hat{v}(0) \end{bmatrix} + \begin{bmatrix} 10 \\ -25 \end{bmatrix} 100 = \begin{bmatrix} -10 & -1 \\ 25 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix} + \begin{bmatrix} 1000 \\ -2500 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} \dot{\hat{d}}(0) \\ \dot{\hat{v}}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

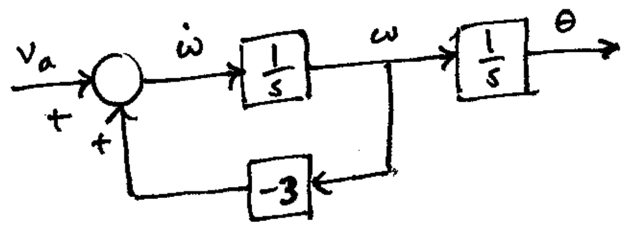
i.e., the "slopes" of the plots for $\hat{d}$ and $\hat{v}$ at $t = 0$ are zero. This is the reason behind the discrepancy between $\hat{d}(t)$ and $d(t)$ at small time. If $\hat{v}(0)$ had been set to the correct value $\hat{v}(0) = 10$, then $\dot{\hat{d}}(0) = -10$ would have lead to the correct initial slope.

PK

(3)



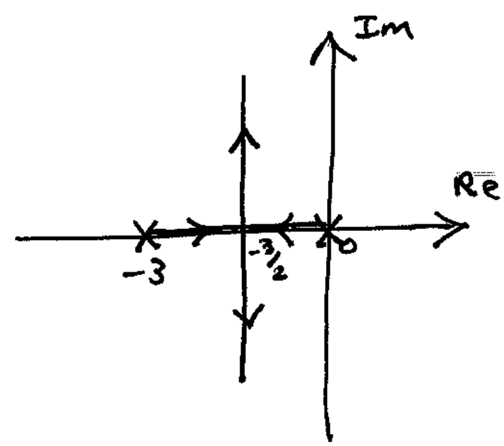
(we redraw the block diagram) $\Leftrightarrow$



From the figure above we get

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_a$$

When $K(s)$ is just a constant gain the root-locus is as shown on right



If $v_a(t) = -k_1\theta(t) - k_2\omega(t)$ we have for the char. equ.

$$\det(sI - [A - bk^T]) = \det\begin{bmatrix} s & -1 \\ k_1 & s+3+k_2 \end{bmatrix} = s^2 + (3+k_2)s + k_1 = 0$$

On the other hand the desired char. equ. is

$$(s+5+5j)(s+5-5j) = s^2 + 10s + 50 = 0$$

$$\Rightarrow \begin{cases} 3+k_2 = 10 \\ k_1 = 50 \end{cases} \Rightarrow \begin{cases} k_1 = 50 \\ k_2 = 7 \end{cases} \Rightarrow k^T = [50 \;\; 7].$$

The observer dynamics are governed by

$$\dot{\hat{x}} = (A - \ell c^T)\tilde{x} + bu + \ell y$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} [1 \quad 0] \right) \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_a + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} y$$

Note that we have taken $c^T = [1 \quad 0]$ because the output is the shaft angle θ. To find ℓ we write the char. equ. of the observer

$$\det(sI - [A - \ell c^T]) = \det \begin{bmatrix} s+\ell_1 & -1 \\ \ell_2 & s+3 \end{bmatrix} = s^2 + (3+\ell_1)s + 3\ell_1 + \ell_2 = 0$$

On the other hand the desired char. equ. is

$$(s+20+20j)(s+20-20j) = s^2 + 40s + 800 = 0$$

$$\Rightarrow \begin{cases} 3+\ell_1 = 40 \\ 3\ell_1 + \ell_2 = 800 \end{cases} \Rightarrow \begin{cases} \ell_1 = 37 \\ \ell_2 = 689 \end{cases} \Rightarrow \ell = \begin{bmatrix} 37 \\ 689 \end{bmatrix}.$$

Therefore the observer equations are

$$\dot{\hat{x}} = \begin{bmatrix} -37 & 1 \\ -689 & -3 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 37 \\ 689 \end{bmatrix} y$$

The transfer function of the controller that results from combining state observation & feedback is

$$K(s) = k^T (sI - A + bk^T + \ell c^T)^{-1} \ell$$

$$= [50 \quad 7] \begin{bmatrix} s+37 & -1 \\ 739 & s+10 \end{bmatrix}^{-1} \begin{bmatrix} 37 \\ 689 \end{bmatrix} = \frac{6673\left(s + \frac{40000}{6673}\right)}{s^2 + 47s + 1109}$$

The transfer function of the entire system can be either found from the formula given in class, or, by using the closed-loop block diagram to show that $\frac{\theta_c}{\theta} = \frac{KG}{1+KG}$.

$$\frac{\theta}{\theta_c} = \frac{6673\left(s+\frac{40000}{6673}\right)}{(s^2+47s+1109)s(s+3)+6673s+40000}$$

$$= \frac{6673\left(s+\frac{40000}{6673}\right)}{s^4+50s^3+1250s^2+10000s+40000}$$

$$= \frac{6673\left(s+\frac{40000}{6673}\right)}{(s^2+10s+50)(s^2+40s+800)} =: G_{cl}(s)$$

Note that the closed-loop poles are made of the poles of the observer and the poles of the state feedback.

This system tracks step functions, because

$$G_{cl}(0) = 1. \quad \text{(i.e., unit D.C. gain)}$$

Another way to see this is to notice that the forward loop contains a "$\frac{1}{s}$" term. Therefore

$$G_{cl} = \frac{K\frac{1}{s(s+3)}}{1+K\frac{1}{s(s+3)}} = \frac{K}{s(s+3)+K}$$

$$\Rightarrow G_{cl}(s=0) = \frac{K(0)}{0+K(0)} = 1.$$

the "$\frac{1}{s}$" term guarantees tracking of step commands