Solutions to Homework 8

(1)

It is easy to show that

$$\lambda_1 = -2 \qquad \lambda_2 = 1$$
$$p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad p_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$\rightarrow \lambda_2 = 1 > 0$ system is <u>unstable</u>.

We have

$$T = [p_1 \;\; p_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \rightarrow \quad T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

And therefore

$$T^{-1}b = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$c^T T = \left[\tfrac{1}{2} \;\; \tfrac{1}{2}\right]\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [1 \;\; 0]$$

So the diagonalized system is

$$\dot{z} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$
$$y = [1 \;\; 0]\, z$$

Clearly mode $\lambda_2 = 1$ is neither controllable nor observable, and mode $\lambda_1 = -2$ is both controllable and observable.

$$G(s) = [1 \;\; 0]\begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s-1} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{2}{s+2}$$

mode $\lambda_2 = -1$ does not appear in transfer function.

(2)

$$G_1(s) = \frac{s+1}{s(s+3)} = \frac{s+1}{s^2+3s+0} \rightarrow A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix},\ b_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ c_1^T = [1 \quad 1]$$

$$G_2(s) = \frac{k}{s+a} \rightarrow A_2 = -a,\ b_2 = 1,\ c_2 = k.$$

Therefore the state-space matrices of the overall feedback interconnection are

$$A = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} & -\begin{bmatrix} 0 \\ 1 \end{bmatrix} k \\ 1 \cdot [1 \quad 1] & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & -k \\ 1 & 1 & -a \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = [c_1 \quad 0] = [[1 \quad 1] \quad 0] = [1 \quad 1 \quad 0].$$

The controllability matrix is thus

$$M_c = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 9-k \\ 0 & 1 & -2-a \end{bmatrix} \xRightarrow{a=1} M_c = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 9-k \\ 0 & 1 & -3 \end{bmatrix}$$

Clearly for $a=1$ the matrix M_c is not full rank, because

$$k \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -3 \\ 9-k \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The observability matrix is

$$M_o = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -k \\ -k & 6-k & 2k+ka \end{bmatrix} \xrightarrow{a=1} M_o = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -k \\ -k & 6-k & 3k \end{bmatrix}$$

It is clear that for $a=1$ the matrix M_o is not full rank, because

$$-1 \cdot \begin{bmatrix} 1 \\ 0 \\ -k \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ -2 \\ 6-k \end{bmatrix} - \frac{2}{k} \cdot \begin{bmatrix} 0 \\ -k \\ 3k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3

From the picture, we get (picture on next page)

$$\left.\begin{aligned} \dot{x}_1 &= -4x_1 - 5(u - x_2) \\ &= -4x_1 + 5x_2 - 5u \\ \\ \dot{x}_2 &= x_2 + y \\ &= x_2 + x_1 + (u - x_2) \\ &= x_1 + u \end{aligned}\right\} \quad \begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \end{aligned}$$

$$M_c = \begin{bmatrix} -5 & 25 \\ 1 & -5 \end{bmatrix}, \quad M_o = \begin{bmatrix} 1 & -1 \\ -5 & 5 \end{bmatrix}. \longrightarrow \text{rank } M_c = \text{rank } M_o = 1.$$

The ranks of M_c and M_o are both less than 2. Thus the ~~system~~ realization is neither controllable nor observable.

$$G(s) = \begin{bmatrix} -1 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} -5 \\ 1 \end{bmatrix} + 1$$

$$= \frac{1}{s^2+4s-5} \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} s & 5 \\ 1 & s+4 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix} + 1$$

$$= \frac{-6\cancel{(s-1)}}{(s+5)\cancel{(s-1)}} + 1 = \frac{s-1}{s+5}$$