Solutions to Homework 7

①

$$M_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 & -5 & 9 & 25 \\ 1 & -1 & -3 & 5 & 9 & -25 \\ 1 & 0 & -3 & 0 & 9 & 0 \end{bmatrix}$$

There are only two linearly independent vectors in the columns of M_c; these are $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. All other columns are basically some multiple of one of these vectors. Therefore

$$\text{rank}(M_c) = 2 .$$

Since the dimension of the system is $n=3$, and $\text{rank}(M_c) = 2 < 3$, then the system is not controllable.

②

$$M_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} b_1 & ab_1 & a^2b_1 \\ b_2 & ab_2 & a^2b_2 \\ b_3 & ab_3 & a^2b_3 \end{bmatrix}$$

All columns of M_c are multiples of the vector $v = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Therefore

$$\text{rank}(M_c) = 1 .$$

Since the dimension of the system is $n=3$, and $\text{rank}(M_c) = 1 < 3$, then the system is not controllable.

The column space of M_c is the space of all vectors aligned with $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, i.e., $\mathcal{C} = \left\{ \alpha \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} ; \alpha \in \mathbb{R} \right\}$.

P.T.O.

(3)

$$x_1 = \theta_1$$
$$x_2 = \theta_2$$
$$x_3 = \dot{x}_1 = \dot{\theta}_1 \longrightarrow \ddot{\theta}_1 = \dot{x}_3$$
$$x_4 = \dot{x}_2 = \dot{\theta}_2 \longrightarrow \ddot{\theta}_2 = \dot{x}_4$$

The equations of motion become

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = -\frac{ka^2}{m\ell^2}x_1 + \frac{ka^2}{m\ell^2}x_2 - \frac{g}{\ell}x_1 - \frac{1}{m\ell^2}u$$
$$\dot{x}_4 = -\frac{ka^2}{m\ell^2}x_2 + \frac{ka^2}{m\ell^2}x_1 - \frac{g}{\ell}x_2 + \frac{1}{m\ell^2}u$$

or in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\kappa-\gamma & \kappa & 0 & 0\\ \kappa & -\kappa-\gamma & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ -\mu\\ \mu\end{bmatrix}u$$

where $\kappa := \frac{ka^2}{m\ell^2}$, $\gamma := \frac{g}{\ell}$, $\mu := \frac{1}{m\ell^2}$.

$$M_c = \begin{bmatrix}0 & -\mu & 0 & (\kappa+\gamma)\mu + \kappa\mu\\ 0 & \mu & 0 & -\kappa\mu-(\kappa+\gamma)\mu\\ -\mu & 0 & (\kappa+\gamma)\mu+\kappa\mu & 0\\ \mu & 0 & -\kappa\mu-(\kappa+\gamma)\mu & 0\end{bmatrix}$$

There are only two linearly independent vectors in the columns of M_c; these are aligned with $v_1 = \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$. All columns are multiples of v_1, v_2.

Therefore

$$\text{rank}\,(M_c) = 2.$$

Since $n=4$ and $\text{rank}(M_c) = 2 < 4$, the system is not controllable.

The column space of M_c is

$$\mathcal{C} = \left\{ \alpha_1 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\alpha_2 \\ \alpha_2 \\ -\alpha_1 \\ \alpha_1 \end{bmatrix} ; \ \alpha_1, \alpha_2 \in \mathbb{R} \right\}.$$

Note that $\mathcal{C}$ is a plane in $\mathbb{R}^4$ space.

To find the transfer function, we note that since we are interested in θ_1, we choose the C matrix such that

$$y = Cx = \theta_1$$

Recall that

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Therefore

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

So we form

$$G(s) = C(sI-A)^{-1}B$$

which, after some calculations, gives

$$G(s) = \frac{(\gamma + 2\kappa)\mu^2}{s\left(s - \mu(1+\gamma+2\kappa)\right)}.$$

If $m=1$, $l=1$, $a=\frac{1}{2}$, $g=10$, then

$$\kappa = 1, \quad \gamma = 10, \quad \mu = 1 .$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11 & 1 & 0 & 0 \\ 1 & -11 & 0 & 0 \end{bmatrix}$$

The eigenvalues of A are $\begin{cases} s_1 = 3.464\, i \\ s_2 = -3.464\, i \\ s_3 = 3.162\, i \\ s_4 = -3.162\, i \end{cases}$

which are all purely imaginary. Therefore the system is "marginally stable".

④

By definition of C_k, we have

$$C_k = \left[b \;\; Ab \;\; \cdots \;\; A^{k-1} b \right]$$

$$C_{k+1} = \left[b \;\; Ab \;\; \cdots \;\; A^{k-1} b \;\; A^k b \right] = \left[C_k \;\; A^k b \right].$$

From the assumption $\operatorname{rank} C_{k+1} = \operatorname{rank} C_k$ it follows that $A^k b$ can be written as a linear combination of the columns of C_k, i.e.,

$$A^k b = \alpha_0 b + \alpha_1 Ab + \cdots + \alpha_{k-1} A^{k-1} b \qquad (*)$$

for some real numbers $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$. If we multiply the above equation by A, we get

$$A^{k+1} b = A(A^k b) = \alpha_0 Ab + \alpha_1 A^2 b + \cdots + \alpha_{k-1} A^k b . \quad (**)$$

Notice that the term $A^k b$ appears in the expression for $A^{k+1}b$. Therefore we use equation (*) to replace $A^k b$ in equation (**) with lower powers of $A^i b$, $i=0,\dots,k-1$, which gives

$$A^{k+1}b = \alpha_0 Ab + \alpha_1 A^2 b + \cdots + \alpha_{k-2}A^{k-1}b$$
$$+ \alpha_{k-1}\left(\alpha_0 b + \alpha_1 Ab + \cdots + \alpha_{k-1}A^{k-1}b\right)$$
$$=: \beta_0 b + \beta_1 Ab + \cdots + \beta_{k-1}A^{k-1}b\,,$$

for some real numbers $\beta_0, \beta_1, \dots, \beta_{k-1}$. (Notice that it is possible to find each β_i in terms of the α_j. But this is <u>not</u> important to us. What matters is that we can find <u>"some"</u> set of numbers β_i, $i=1,\dots,k-1$ such that

$$A^{k+1}b = \beta_0 b + \beta_1 Ab + \cdots + \beta_{k-1}A^{k-1}b.\ )$$

This means that if we form C_{k+2}

$$C_{k+2} = \left[C_k \;\; A^k b \;\; A^{k+1}b \right]$$

then both $A^k b$ and $A^{k+1}b$ can be written as a linear combination of the columns of C_k. Therefore

$$\text{rank } C_{k+2} = \text{rank } C_k\,.$$

The same procedure can be applied to C_{k+i} for any $i \geqslant 1$. Therefore rank C_{k+i} = rank C_k for all $i \geqslant 1$.