Solutions to Homework 6

①

There are many ways to solve this problem.

METHOD 1:

$$e^{At} = e^{T\Lambda T^{-1}t} = Te^{\Lambda t}T^{-1} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}$$

$$\rightarrow e^{At}x_0 = q_1^T x_0 e^{\lambda_1 t} p_1 + q_2^T x_0 e^{\lambda_2 t} p_2 = x(t)$$

From the problem statement, we have

$$x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \Rightarrow \begin{cases} \lambda_1 = -1 \\ p_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}.$$

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} \Rightarrow \begin{cases} \lambda_2 = 1 \\ p_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

$$\Rightarrow T = \begin{bmatrix} p_1 & p_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \rightarrow T^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \rightarrow \begin{cases} q_1^T = [1/2 \;\; -1/2] \\ q_2^T = [1/2 \;\; 1/2] \end{cases}$$

$$\Rightarrow e^{At} = Te^{\Lambda t}T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}.$$

$$\& \quad A = T\Lambda T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

METHOD 2:

Suppose $e^{At} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We know that $x(t) = e^{At} x_0$.

$$x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow x(t) = e^{At} x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \rightarrow \begin{cases} a_{11} - a_{12} = e^{-t} \\ a_{21} - a_{22} = -e^{-t} \end{cases}$$

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow x(t) = e^{At} x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} \rightarrow \begin{cases} a_{11} + a_{12} = e^{t} \\ a_{21} + a_{22} = e^{t} \end{cases}$$

Therefore we have 4 equations and 4 unknowns. We can thus find $a_{11}, a_{12}, a_{21}, a_{22}$

$$\begin{cases} a_{11} = \dfrac{e^{t} + e^{-t}}{2} \\ a_{12} = \dfrac{e^{t} - e^{-t}}{2} \\ a_{21} = \dfrac{e^{t} - e^{-t}}{2} \\ a_{22} = \dfrac{e^{t} + e^{-t}}{2} \end{cases} \rightarrow e^{At} = \frac{1}{2} \begin{bmatrix} e^{t} + e^{-t} & e^{t} - e^{-t} \\ e^{t} - e^{-t} & e^{t} + e^{-t} \end{bmatrix}.$$

In particular, this means that $(sI - A)^{-1} = \mathcal{L}\{e^{At}\}$

$$= \frac{1}{(s+1)(s-1)} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}$$

To find A, assume $A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$. Then

$$(sI - A)^{-1} = \begin{bmatrix} s - \alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & s - \alpha_{22} \end{bmatrix}^{-1} = \frac{1}{(s-\alpha_{11})(s-\alpha_{22}) - \alpha_{12}\alpha_{21}} \begin{bmatrix} s - \alpha_{22} & \alpha_{21} \\ \alpha_{12} & s - \alpha_{11} \end{bmatrix}$$

Therefore

$$\frac{1}{s^2-(\alpha_{11}+\alpha_{22})s+\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}\begin{bmatrix} s-\alpha_{22} & \alpha_{21} \\ \alpha_{12} & s-\alpha_{11}\end{bmatrix} = \frac{1}{s^2-1}\begin{bmatrix} s & 1 \\ 1 & s\end{bmatrix}$$

This gives

$$\begin{cases} \alpha_{11}=0 \\ \alpha_{12}=1 \\ \alpha_{21}=1 \\ \alpha_{22}=0 \end{cases} \longrightarrow \quad A=\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}.$$

Thus we have solved parts (a) & (c) of the problem.

(b) $$x(t) = e^{At}x(0)$$

$$= \frac{1}{2}\begin{bmatrix} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t}\end{bmatrix}\begin{bmatrix} 2 \\ 0\end{bmatrix} = \begin{bmatrix} e^t+e^{-t} \\ e^t-e^{-t}\end{bmatrix}$$

(d) $$x(t) = \underbrace{e^{At}x(0)}_{\text{affect of initial cond.}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{affect of input.}}$$

$$\cancel{e^{At}x(0) = \frac{1}{2}\begin{bmatrix} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t}\end{bmatrix}\begin{bmatrix} 1 \\ 0\end{bmatrix}}$$

$$y(t) = Cx(t) + \overset{0}{\cancel{Du(t)}}$$

$$= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau.$$

$$= Ce^{At}x(0) + Ce^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau.$$

$$c e^{At} x(0) = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} \left(e^{t} - e^{-t} \right).$$

$$\int_0^t e^{-A\tau} B u(\tau)\, d\tau = \int_0^t \frac{1}{2} \begin{bmatrix} e^{-\tau}+e^{+\tau} & e^{-\tau}-e^{+\tau} \\ e^{-\tau}-e^{+\tau} & e^{-\tau}+e^{+\tau} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cdot 1 \cdot d\tau$$

$$= \int_0^t -\frac{1}{2} \begin{bmatrix} e^{-\tau} - e^{+\tau} \\ e^{-\tau} + e^{+\tau} \end{bmatrix} d\tau = -\frac{1}{2} \begin{bmatrix} -e^{-\tau} - e^{+\tau} \\ -e^{-\tau} + e^{+\tau} \end{bmatrix} \Bigg|_0^t$$

$$= -\frac{1}{2} \begin{bmatrix} -e^{-t} - e^{+t} + 2 \\ -e^{-t} + e^{+t} - 0 \end{bmatrix}.$$

$$c e^{At} = \frac{1}{2} \begin{bmatrix} e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}.$$

$$c e^{At} \int_0^t e^{-A\tau} B u(\tau)\, d\tau = -\frac{1}{4} \begin{bmatrix} e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix} \begin{bmatrix} -e^{-t} - e^{+t} + 2 \\ -e^{-t} + e^{+t} \end{bmatrix}$$

$$= -\frac{1}{4} \left(-1 - e^{2t} + 2e^{t} + e^{-2t} + \cancel{1} - 2e^{-t} - \cancel{1} + \cancel{e^{2t}} - \cancel{e^{-2t}} + \cancel{1} \right)$$

$$= -\frac{1}{4} \left(+2e^{t} - 2e^{-t} \right) = -\frac{1}{2} \left(e^{t} - e^{-t} \right)$$

$$\Rightarrow c e^{At} x(0) + c e^{At} \int_0^t e^{-A\tau} B u(\tau)\, d\tau = \frac{e^{t}-e^{-t}}{2} - \frac{e^{t}-e^{-t}}{2} = 0$$

②

a- The governing equations for the circuit are

$$\begin{cases} 10\,\dfrac{di_1}{dt} = u(t) - 30(i_1 - i_2) \\[2mm] 9\,\dfrac{di_2}{dt} = 30(i_1 - i_2) - 6\,i_2 \end{cases}$$

Taking i_1, i_2 as the state variables of the system, we get

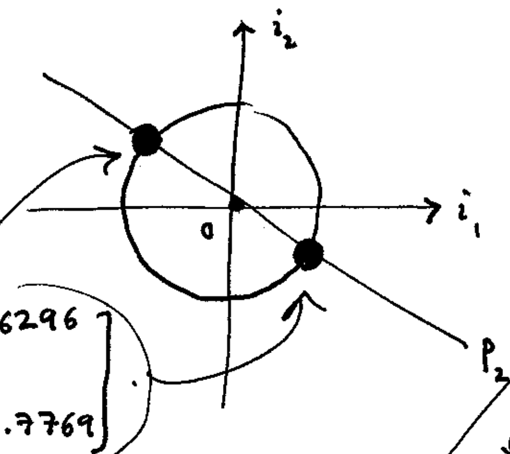
$$\begin{bmatrix} \dot{i}_1 \\ \dot{i}_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ \frac{30}{9} & -4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} u(t).$$

b- We now find the eigenvalues of A:

$$\det(sI - A) = \det \begin{bmatrix} s+3 & -3 \\ -\frac{30}{9} & s+4 \end{bmatrix} = s^2 + 7s + 2 = 0 \longrightarrow \begin{cases} \lambda_1 = -0.2984 \\ \lambda_2 = -6.7016 \end{cases}$$

Clearly $e^{\lambda_2 t} \approx e^{-6.7t}$ goes to zero much faster than $e^{\lambda_1 t} \approx e^{-0.3t}$. Therefore, we choose our initial conditions to reside along the direction of the eigenvector corresponding to the eigenvalue λ_2. You can show that the eigenvector corresponding to λ_2 is $p_2 = \begin{bmatrix} -0.8105 \\ 1 \end{bmatrix}$.

Finally, since the initial condition should have unit length, we divide p_2 by its length. Thus the solution is



$$\begin{bmatrix} i_1(0) \\ i_2(0) \end{bmatrix} = \begin{bmatrix} -0.6296 \\ 0.7769 \end{bmatrix} \text{ or } \begin{bmatrix} 0.6296 \\ -0.7769 \end{bmatrix}.$$

PS

③

$$\int_0^t e^{A\sigma}\,d\sigma = \int_0^t \left(I + A\sigma + \frac{1}{2!}A^2\sigma^2 + \cdots\right) d\sigma$$

$$= I\sigma\Big|_0^t + A\frac{\sigma^2}{2}\Big|_0^t + \frac{1}{2!}A^2\frac{\sigma^3}{3}\Big|_0^t + \cdots$$

$$= It + \frac{1}{2}At^2 + \frac{1}{3!}A^2t^3 + \cdots$$

$$\Downarrow$$

If we define $\int_0^t e^{A\sigma}\,d\sigma =: Y(t)$ we have

$$YA = At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = e^{At} - I$$

$$\rightarrow \quad Y = (e^{At} - I)A^{-1}.$$

The solution of $\dot{x} = Ax + Bu$, $x(0) = x_0$ is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

If u is constant in time, then we can use the above formula to simplify the integral.

$$x(t) = e^{At}x_0 + \left(\int_0^t e^{At}e^{-A\tau}B\,d\tau\right)u$$

$$= e^{At}x_0 + e^{At}\left(\int_0^t e^{-A\tau}d\tau\right)Bu$$

used change of variable $s = -\tau$.

$$= e^{At}x_0 + e^{At}\left(-\int_0^{-t} e^{As}\,ds\right)Bu$$

$$= e^{At}x_0 + e^{At}\left[-(e^{-At} - I)A^{-1}\right]Bu.$$

$$= e^{At}x_0 + (e^{At} - I)A^{-1}Bu.$$

④ We have to show that

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0) \quad , \quad \Phi(t_0,t_0) = I.$$

$$\frac{d}{dt}\Phi(t,t_0) = \frac{d}{dt}\left[e^{-Rt}\,e^{(R+S)(t-t_0)}\,e^{Rt_0}\right]$$

$$= -Re^{-Rt}e^{(R+S)(t-t_0)}e^{Rt_0} + e^{-Rt}(R+S)e^{(R+S)(t-t_0)}e^{Rt_0}$$

$$= \cancel{-Re^{-Rt}e^{(R+S)(t-t_0)}}e^{Rt_0}$$

$$+\cancel{e^{-Rt}Re^{(R+S)(t-t_0)}}e^{Rt_0} + e^{-Rt}S\,e^{(R+S)(t-t_0)}e^{Rt_0}$$

$$= e^{-Rt}S\,e^{(R+S)(t-t_0)}e^{Rt_0}$$

using the fact that $Re^{-Rt} = e^{-Rt}R$ ← (this can be proved using the Taylor series definition of e^{-Rt})

On the other hand

$$A(t)\,\Phi(t,t_0) = e^{-Rt}S\,\overbrace{e^{Rt}\,e^{-Rt}}^{I}\,e^{(R+S)(t-t_0)}e^{Rt_0}$$

$$= e^{-Rt}S\,e^{(R+S)(t-t_0)}e^{Rt_0}$$

which is the same as $\frac{d}{dt}\Phi(t,t_0)$. Furthermore

$$\Phi(t_0,t_0) = e^{-Rt_0}e^{(R+S)(t_0-t_0)}e^{Rt_0}$$

$$= e^{-Rt_0}\,I\,e^{Rt_0} = I.$$