Solutions to Homework 1

(1) The key to solving this problem is understanding that for two general matrices X, Y, we have

$X + Y = Y + X$, $XY \neq YX$, i.e., we can change the order of addition, but not the order of multiplication.

$$(A+B)^2 = (A+B)(A+B) = (A+B)(B+A)$$

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B)$$

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$$

$$\neq A^2 + 2AB + B^2.$$

② Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = 0$.

Now consider a general matrix A, and partition it columnwise

$$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \rightarrow A^T = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix}$$

$$A^T A = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \cdots \\ a_2^T a_1 & \ddots & \vdots \\ \vdots & \cdots & a_n^T a_n \end{bmatrix}.$$

If this matrix is equal to zero, then in particular all of its diagonal entries have to be zero

$$a_1^T a_1 = 0 \rightarrow \|a_1\|_2^2 = 0 \rightarrow a_1 = 0$$

$$\vdots$$

$$a_n^T a_n = 0 \rightarrow \|a_n\|_2^2 = 0 \rightarrow a_n = 0$$

Therefore all columns of A have to equal zero, which means that A is the zero matrix.

(3) $A = [1, 2; 3, 4]$

$x = [5; -2]$

$b = [1; 7]$.

From a mathematical point of view, if $Ax = b$ then $\|Ax - b\|_2 = 0$, and vice versa.

In matlab, the vector norm is given by the "norm" command. So, we type the "norm"

norm (A*x - b)

If this value is zero then we know that $Ax = b$.

4 (a) $X = \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix}$

$$AX = \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

(b) $A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow A\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow A\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow A\begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$

add the left & right hand sides

$$\underbrace{A\left(\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}\right)}_{x} = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}}_{b}$$

$\downarrow$

$$x = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}.$$

(c) $A\begin{bmatrix}0\\0\\1\end{bmatrix}$ = third column of A $= \begin{bmatrix}0\\0\\1\end{bmatrix}$

$A\begin{bmatrix}0\\1\\1\end{bmatrix}$ = sum of second & third columns of $A = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$A\begin{bmatrix}1\\1\\1\end{bmatrix}$ = sum of all three columns of $A = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Suppose $A = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}$

$$\begin{cases} a_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \\ a_2 + a_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} \to a_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} \to a_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix} \\ a_1 + a_2 + a_3 = \begin{bmatrix}1\\0\\0\end{bmatrix} \to a_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} \to a_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix}. \end{cases}$$

$\downarrow$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

5 By definition, $x \in N(A)$. Now partition A row wise

$$A = \begin{bmatrix} - a_1^T - \\ - a_2^T - \end{bmatrix} \rightarrow a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

$$Ax = 0 \rightarrow \begin{bmatrix} - a_1^T - \\ - a_2^T - \end{bmatrix} x = 0 \rightarrow \begin{cases} a_1^T x = 0 \\ a_2^T x = 0 \end{cases}$$

This implies that any vector x that is a solution must be orthogonal to both a_1 & a_2.

On the other hand, the set of all $x \in \mathbb{R}^3$, orthogonal to $y \in \mathbb{R}^3$, defines a plane. Therefore

$$\begin{cases} a_1^T x = 0 \rightarrow x \text{ belongs to the plane } P_1 \text{ orthogonal to } a_1 \\ a_2^T x = 0 \rightarrow x \text{ belongs to the plane } P_2 \text{ orthogonal to } a_2 \end{cases}$$

→ if x simultaneously satisfies these two equations then it must belong to the intersection of the two planes P_1 & P_2. And the intersection of two distinct planes is a line.

6 It is easy to see that the columns of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ span all of $\mathbb{R}^3$: For example, I can combine the first two columns to get

$$v = a_2 - a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and combine the last two columns to get

$$w = a_3 - a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Together with the first column

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

I obtain the set of vectors $\left\{ \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}^{a_1}, \overbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}^{v}, \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{w} \right\},$

which clearly spans the whole space.

Thus $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = b$ has a solution for any b.

On the other hand, no matter how I combine the columns of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, I can never create a vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with $b_3 \neq 0$. But if $b_3 = 0$ then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = b$ always has a solution.

⑦ $v_1 = w_2 - w_3$, $v_2 = w_3 - w_1$, $v_3 = w_1 - w_2$

$$1v_1 + 1v_2 + 1v_3 = (\cancel{w_2} - \cancel{w_3}) + (\cancel{w_3} - \cancel{w_1}) + (\cancel{w_1} - \cancel{w_2})$$

$$= 0.$$

Therefore v_1, v_2, v_3 are linearly dependent.