1. Consider an LTI system that has the following properties. When the input is zero and the initial state is given by

$$x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

then

$$x(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{e}^{-t}.$$

And when the input is zero and the initial state is given by

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

then

$$x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{t}.$$

   (a) Write an explicit expression (not the infinite series) for the matrix exponential e^{At}.

   (b) Suppose that $u(t) = 0$ for all $t \geq 0$, and that

$$x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

   Find $x(t)$ for all $t \geq 0$.

   (c) Find the matrix A.

   (d) Now suppose that

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D = 0,$$

   and

$$u(t) = 1 \ \text{ for all } \ t \geq 0,$$

   and

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

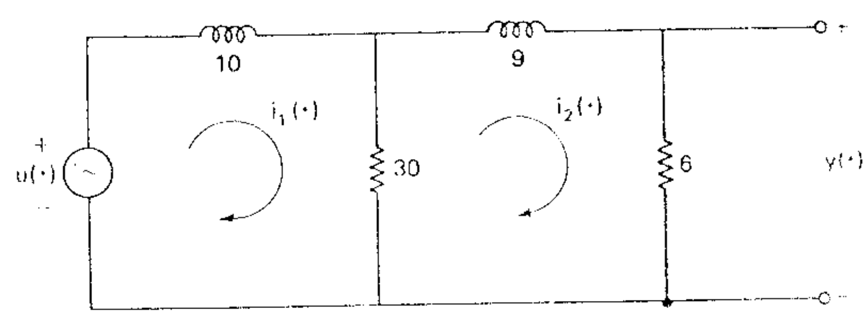
   Then show that the output is zero for all time, i.e.,

$$y(t) = 0 \ \text{ for all } \ t \geq 0.$$

2. Kailath, Exercise 2.5-9:

   (a) Write state equations for the network shown in the figure.

   (b) Assume that the input u is turned off at $t = 0$. Show how to choose $i_1(0)$, $i_2(0)$, subject to the constraint $i_1(0)^2 + i_2(0)^2 = 1$, so that the systems returns to rest at the fastest possible rate. [By 'returning to rest at the fastest possible rate' we mean that the states i_1 and i_2 of the system converge to zero as fast as possible.]



3. Let A be an $n \times n$ invertible matrix. Use the Taylor series definition of the matrix exponential to show that

$$\int_0^t \mathrm{e}^{A\sigma}\, d\sigma = (\mathrm{e}^{At} - I)\, A^{-1}.$$

Using this result, obtain the solution to the linear time-invariant system

$$\dot{x} = A\,x + B\,u, \qquad x(0) = x_0,$$

when the input u is a *constant* $r \times 1$ vector (and B is an $n \times r$ matrix).

4. Consider the zero-input linear time-varying system

$$\dot{x}(t) = \mathrm{e}^{-Rt}\, S\, \mathrm{e}^{Rt}\, x(t),$$

with R and S constant square matrices. Show that the state transition matrix of the system is

$$\Phi(t, t_0) = \mathrm{e}^{-Rt}\, \mathrm{e}^{(R+S)(t-t_0)}\, \mathrm{e}^{Rt_0},$$

i.e., $x(t) = \Phi(t, t_0)\, x(t_0) = \mathrm{e}^{-Rt}\, \mathrm{e}^{(R+S)(t-t_0)}\, \mathrm{e}^{Rt_0}\, x(t_0)$.
Hint: Just use the definition of the state transition matrix, $\frac{d}{dt}\Phi(t, t_0) = A(t)\,\Phi(t, t_0)$ and $\Phi(t_0, t_0) = I$.