1. Kailath, Exercise 2.2-3: An important problem in simulation is that of scaling, or choosing the correct units for the variables. Suppose we have a realization given by A, b, c^T, with the three-dimensional state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Suppose we now change to the variables

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},$$

where $z_1 = k_1 x_1$, $z_2 = k_2 x_2$, and $z_3 = k_3 x_3$, and we let $\dot{z} = Fz + gu$, $y = h^T z$.

Write out the matrices F, g, h^T in terms of the elements of A, b, c^T and the scale factors k_1, k_2, k_3 (i.e., show the elements f_{ij} of F in terms of the elements a_{ij} of A and the scale factors k_1, k_2, k_3, and similarly for g and h^T).

Hint: Note that the relation between x and z an be written as $x = Tz$, where T is a matrix. Find T, then substitute Tz for x in the system equations, and simplify.

2. For an LTI system with nonzero initial conditions $x(0) = x_0$ and zero input, the output is given by $y(t) = Ce^{At}x_0$; this expression uses the matrix exponential e^M, which for the square matrix M is defined as

$$e^M = I + \frac{M}{1!} + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots .$$

Show that the output $y(t)$ remains invariant under any similarity transformation of the state, $x(t) = Tz(t)$, where T is an invertible matrix.

3. Using the Taylor series definition of a matrix exponential, find the matrix e^{At} when

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}.$$

4. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}.$$

   (a) Calculate the eigenvalues of A.
   (b) Calculate the eigenvectors of A.
   (c) Determine the similarity transformation T that diagonalizes A and the corresponding diagonal matrix Λ. Verify that your proposed similarity transformation indeed diagonalizes A, i.e., check whether $A = T\Lambda T^{-1}$.
   (d) Compute e^{At} using diagonalization.

5. Suppose λ and v are an eigenvalue-eigenvector pair for the matrix A, $Av = \lambda v$.

   (a) Show that λ^m and v are an eigenvalue-eigenvector pair for the matrix A^m, where m is any positive integer.
   (b) Show that $\lambda + \alpha$ and v are an eigenvalue-eigenvector pair for the matrix $A + \alpha I$, where α is any real number.

6. Kailath, Appendix A.44,45: Suppose the $n \times n$ matrix A has n eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ with n linearly independent associated eigenvectors $\{p_1, \ldots, p_n\}$. Let $P = [p_1, \ldots, p_n]$, $Q = P^{-1}$, and let $\{q_i^T\}$ be the rows of Q.

Convince yourself (you don't have to show this in your solutions) that P^{-1} exists, and that we have

$$q_j^T p_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

[In fact, it can be shown that $A^T q_i = \lambda_i\, q_i$, or equivalently $q_i^T A = \lambda_i\, q_i^T$. The vectors $\{q_1^T, \ldots, q_n^T\}$ are often called the *left* (or *row*) eigenvectors of A in contrast to the *right* (or *column*) eigenvectors $\{p_1, \ldots, p_n\}$.]

Now let A, P, Q be as described above. Show that we can write

1. $A = \sum_{i=1}^{n} \lambda_i\, p_i\, q_i^T = P\Lambda\, Q$, where $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$.
2. $A^2 = \sum_{i=1}^{n} \lambda_i^2\, p_i\, q_i^T = P\Lambda^2 Q$.
3. $f(A) = \sum_{i=1}^{n} f(\lambda_i)\, p_i\, q_i^T = P f(\Lambda)\, Q$, where $f(A)$ is any polynomial in A.

7. Suppose the matrix A is diagonalizable, its powers A^k approach a limit as $k \to \infty$, and

$$A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}.$$

What can you say about the eigenvalues of A? If you are additionally told that $A^2 = \frac{1}{2}(A + A^\infty)$, can you find the eigenvalues of A?