ELE 412 students do not need to hand in problems marked with '$*$'

1. Consider the three points $(x, y) = (1, 1), (2, 1), (3, 2)$ in the xy-plane. It is desired to find the coefficients a_0 and a_1 that describe the line $y = a_1 x + a_0$ such that $e_1^2 + e_2^2 + e_3^2$ is minimized, where e_i is the vertical distance of the ith point from the line. Formulate this as a least-squares problem and find the optimal values of a_0 and a_1.

2. Find the best straight-line fit (in the sense of least-squares) to the measurements

$$\begin{array}{llll} b = 4 \quad \text{at} \quad t = -2, & & b = 3 \quad \text{at} \quad t = -1, \\ b = 1 \quad \text{at} \quad t = 0, & & b = 0 \quad \text{at} \quad t = 2. \end{array}$$

   Then find the projection of $b = [4, 3, 1, 0]^T$ onto the column space of

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

3. This problem is about fitting a model to hourly temperature. You are given a set of temperature measurements

$$y_t, \qquad t = 1, \ldots, N,$$

   taken hourly over one week (so $N = 168$). An expert says that over this week, an appropriate model for the hourly temperature is

$$\hat{y}_t = at + p_t, \qquad t = 1, \ldots, N,$$

   where a is a scalar and

$$p_t, \qquad t = 1, \ldots, N,$$

   are N numbers that satisfy $p_{t+24} = p_t$ for $t = 1, \ldots, N-24$ (i.e., the numbers p_t repeat with the period of 24 hours so that $p_{25} = p_1$, $p_{26} = p_2$, etc.). We can interpret a as the warming ($a > 0$) or cooling ($a < 0$) trend over the week, and p_t as the daily temperature oscillations.

   Explain how to find a and p_t, $t = 1, \ldots, N$, such that the error vector

$$e = y - \hat{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix}$$

   has minimum length.

$*$4. In solving the least squares problem in class, we *assumed* that $A^T A$ is invertible. Prove that $A^T A$ is indeed invertible, i.e., using the assumption made in the least squares problem that the columns of A are linearly independent, prove that $(A^T A)^{-1}$ exists.

   Hint: (a) Show that the two matrices A and $A^T A$ have the same null space, i.e., $\mathcal{N}(A) = \mathcal{N}(A^T A)$; (b) Use the assumption that the columns of A are linearly independent and therefore its null space contains only the vector zero; (c) Use the fact that if the null space of a square matrix contains only the zero vector, then that matrix is invertible.

5. There is a special command for computing the least-squares solution in MATLAB: Assuming the columns of A are linearly independent, the unique x that minimizes $\|Ax - y\|$ can be found in MATLAB by using the 'backslash' operator `A\y`. Of course, another way to compute the same solution is to use the formula `inv(A'*A)*A'*y` derived in class.

   Choose an A matrix in $\mathbb{R}^{3\times 2}$ with linearly independent columns, a y vector in $\mathbb{R}^3$ that is not in the column space of A, and find the least-squares solution in MATLAB using both of these methods.