1. Use Lagrange multipliers to solve the optimization problem

$$\begin{array}{ll} \text{maximize} & xyz \\ \text{subject to} & xy + yz + zx = \dfrac{c}{2}, \end{array}$$

where $c > 0$ is a given constant.

This problem can be interpreted as constructing a cardboard box of maximum volume given a fixed area of cardboard c.

2. At different times during the semester (e.g., when computing the induced 2-norm of a diagonal matrix), we have had to solve the problem

$$\begin{array}{ll} \text{maximize} & \|Ax\|_2^2 \\ \text{subject to} & \|x\|_2^2 = 1, \end{array} \qquad \text{or equivalently,} \qquad \begin{array}{ll} \text{minimize} & -\|Ax\|_2^2 \\ \text{subject to} & \|x\|_2^2 = 1, \end{array}$$

where A is a diagonal matrix. Let us consider the problem on the right.

   (a) Define the Lagrangian and find the first-order necessary conditions for optimality.

   (b) Show that there are n possible solutions to the equations found in part (a).

   (c) Of the n possible solutions found in part (b), which one results in the smallest value of the objective?

   (d) Show that the Hessian of the Lagrangian is a positive semidefinite matrix when evaluated at the solution found in part (c).

3. Is the set

$$S = \{x \in \mathbb{R}^n \mid x \succeq 0,\ \mathbb{1}^T x = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\},$$

where $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$, a polyhedron? Explain.

4. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & J(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

First make a sketch of the feasible set. Then for each of the following objective functions, using simple graphical arguments based on the level curves of J, give the optimal value $p^* = J(x^*)$ and the optimal point x^*. (In some cases the optimal value may be achieved by a set of points rather than a single point; in such cases determine the optimal *set*.)

   (a) $J(x_1, x_2) = x_1 + x_2$.

   (b) $J(x_1, x_2) = -x_1 - x_2$.

   (c) $J(x_1, x_2) = x_1$.

   (d) $J(x_1, x_2) = \max\{x_1, x_2\}$.

5. Formulate the following problems as linear programs.

   (a) Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$.

   (b) Minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$.

   (c) Minimize $\|Ax - b\|_1 + \|x\|_\infty$.

In each problem, assume $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given.