On Stability and the Spectrum Determined Growth Condition for Spatially Periodic Systems

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***Abstract*— We consider distributed parameter systems where the underlying dynamics are spatially periodic on the real line. We examine the problem of exponential stability, namely whether the semigroup e^{At} decays exponentially in time. It is known that for distributed systems the condition that the spectrum of A belong to the open left-half plane is, in general, not sufficient for exponential stability. Those systems for which this condition *is* sufficient are said to satisfy the Spectrum Determined Growth Condition (SDGC). In this work we separate A into a spatially invariant operator and a spatially periodic operator. We find conditions for the spatially invariant part to satisfy the SDGC, and show that the SDGC remains satisfied under the addition of a spatially periodic operator if this operator is 'weak' enough relative to the spatially invariant one. A similar method is used to derive conditions which guarantee that A has left-half plane spectrum and thus establish exponential stability.**

I. Introduction

In linear systems theory assessing exponential stability is of particular interest. For finite-dimensional systems (i.e., systems with finite-dimensional state-space) exponential decay of $\|e^{At}\|$ is guaranteed if the spectrum of the A-matrix lies inside the open left-half of the complex plane (open LHP). The situation is much more complicated however in the case of infinite-dimensional systems. For example, it is possible that the spectrum of the A-operator of such a system lie inside the open LHP and yet $\|e^{At}\|$ actually grows exponentially [1]–[3]. In such cases it is said that the *spectrum-determined growth condition* is not satisfied [3].

Yet there exists quite a wide range of infinite-dimensional systems for which the spectrum-determined growth condition *is* satisfied. These include (but are not limited to) systems for which the A-operator is *sectorial* (also known as an operator which generates a *holomorphic* or *analytic* semigroup) [4]–[6] or is a *Riesz-spectral* operator [7].

In this work we focus on sectorial operators. Thus to establish exponential stability of a system, one line of attack would be to show simultaneously that (i) A is sectorial *and* (ii) the spectrum of A lies in the open LHP. But this still does not make the problem trivial. In fact proving that an infinite-dimensional operator is sectorial, and then finding its spectrum, can in general be extremely difficult.

In this paper we will be dealing with a class of *spatially periodic* systems on the real line. These are systems for which the system-operators A, B, and C are spatially periodic (i.e., they commute only with spatial translations of size equal to some real scalar $\mathrm{X} > 0$ called the period) [8], [9]. We consider the A-operator as the sum of a spatially invariant [10] operator A^{o} and a spatially periodic operator $\epsilon\, E$, where ϵ is a complex scalar. Our aim is to find conditions under which this system is exponentially stable.

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To show (i) and (ii) we take an indirect route. We first find conditions on the spatially invariant operator A^{o} such that (i) and (ii) are satisfied. We then show that (i) and (ii) will *remain* satisfied if the spatially periodic operator E is "weaker" than A^{o} in a sense that we describe and if ϵ is small enough. The reason for this indirect approach is that (i) and (ii) are much easier to check for a spatially invariant operator than they are for a spatially periodic one. All conditions we derive are in the Fourier domain and can be checked *pointwise* in the spatial-frequency variable $k \in \mathbb{R}$.

Our presentation is organized as follows. We describe the problem set up in Section II. Section III deals with general notions of the spectrum and sectorial operators. Section IV contains the main contributions of the paper and is divided into two parts; the first part deals with condition (i) described above, and the second part with condition (ii). Conclusions and future directions are given in Section V. Most proofs and technical material are relegated to the Appendix at the end of the paper.

Notation: We use $k \in \mathbb{R}$ to characterize the spatial-frequency variable, also known as the *wave-number*. $\Sigma(T)$ is the spectrum of the operator T, and $\Sigma_{\mathrm{p}}(T)$ its point spectrum, and $\rho(T)$ its resolvent set. To avoid clutter, we do not index norms on different function/operator spaces. We use $\|\cdot\|$ to denote both function and (induced) operator norms on infinite-dimensional spaces and the Euclidean norm for finite-dimensional vectors and matrices; the difference will be clear from the context. $\mathbb{C}^-$ denotes the open left-half of the complex plane, and $j := \sqrt{-1}$. $\overline{\mathfrak{S}}$ is the closure of the set $\mathfrak{S} \subset \mathbb{C}$. We use the same notation for a spatially invariant operator and its Fourier symbol.

II. Problem Setup

Let us consider a system of the form

$$\begin{aligned} \partial_t\, \psi(t,x) \;&= A\, \psi(t,x) \,+\, B\, u(t,x) \\ &= \left(A^{\mathrm{o}} \,+\, B^{\mathrm{o}}\, \epsilon\, F\, C^{\mathrm{o}}\right) \psi(t,x) \,+\, B\, u(t,x), \\ y(t,x) \;&= C\, \psi(t,x), \end{aligned} \qquad (1)$$

where $t \in [0,\infty)$ and $x \in \mathbb{R}$ with the following assumptions. The (possibly unbounded) operators A^{o}, B^{o}, C^{o} are spatially invariant, and the bounded operators B, C are spatially periodic with period $\mathrm{X} = 2\pi/\Omega$. $F(x) = 2L\cos(\Omega x)$ with L a constant matrix, and ϵ is a complex scalar. A^{o}, B^{o}, C^{o} and

$E := B^{\rm o}\, F\, C^{\rm o}$ are all defined on a common dense domain $\mathscr{D} \subset L^2(\mathbb{R})$. $A = A^{\rm o} + \epsilon E$ is closed and generates a *strongly continuous* semigroup (C_0-semigroup), e^{At} [7]. We will refer to A as the *infinitesimal generator* of the system. u, y, and ψ are the spatio-temporal input, output, and state of the system, respectively, and belong to $L^2(\mathbb{R})$ for all t.

Remark 1: We point out that since $\mathscr{D} \subset L^2(\mathbb{R})$, all functions $\phi \in \mathscr{D}$ are such that $\phi(\pm\infty) = 0$. Thus $\mathscr{D}$ contains only information about the smoothness of the functions $\phi \in \mathscr{D}$. ∎

Comment on Notation: We use the same notation for a spatially invariant operator and its Fourier symbol. For example, we use $B^{\rm o}(k) = jk$ to denote the Fourier symbol of the spatially invariant operator $B^{\rm o} = \partial_x$. ∎

As shown in [8], [9], [11], system (1) can be represented in the (spatial) frequency domain by the family of systems

$$\begin{aligned} \partial_t\, \psi_\theta(t) \; &= \big(\mathcal{A}^{\rm o}_\theta \,+\, \epsilon\, \mathcal{B}^{\rm o}_\theta\, \mathcal{F}\, \mathcal{C}^{\rm o}_\theta\big)\, \psi_\theta(t) \,+\, \mathcal{B}_\theta\, u_\theta(t) \\ &= \big(\mathcal{A}^{\rm o}_\theta \,+\, \epsilon\, \mathcal{E}_\theta\big)\, \psi_\theta(t) \,+\, \mathcal{B}_\theta\, u_\theta(t), \qquad (2) \\ y_\theta(t) \; &= \mathcal{C}_\theta\, \psi_\theta(t), \end{aligned}$$

parameterized by $\theta \in [0, \Omega)$, where $\mathcal{A}^{\rm o}_\theta$, $\mathcal{B}^{\rm o}_\theta$, $\mathcal{C}^{\rm o}_\theta$, $\mathcal{B}_\theta$, $\mathcal{C}_\theta$, and $\mathcal{F}$ are operators on ℓ^2. u_θ, y_θ, and ψ_θ belong to ℓ^2 for all t. We have

$$\mathcal{A}^{\rm o}_\theta \;=\; \begin{bmatrix} \ddots & & \\ & A_0(\theta_n) & \\ & & \ddots \end{bmatrix},$$

$$\mathcal{B}^{\rm o}_\theta \;=\; \begin{bmatrix} \ddots & & \\ & B^{\rm o}(\theta_n) & \\ & & \ddots \end{bmatrix}, \quad \mathcal{C}^{\rm o}_\theta \;=\; \begin{bmatrix} \ddots & & \\ & C^{\rm o}(\theta_n) & \\ & & \ddots \end{bmatrix},$$

$$\mathcal{E}_\theta :=\; \mathcal{B}^{\rm o}_\theta\, \mathcal{F}\, \mathcal{C}^{\rm o}_\theta \;=\; \begin{bmatrix} \ddots & \ddots & & \\ \ddots & 0 & A_{-1}(\theta_n) & \\ & A_1(\theta_n) & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \qquad (3)$$

with $\theta_n := \theta + \Omega n$, $n \in \mathbb{Z}$, and

$$A_1(\,\cdot\,) \;:=\; B^{\rm o}(\,\cdot\,)\, L\, C^{\rm o}(\,\cdot\, - \Omega), \qquad (4)$$
$$A_{-1}(\,\cdot\,) \;:=\; B^{\rm o}(\,\cdot\,)\, L\, C^{\rm o}(\,\cdot\, + \Omega), \qquad (5)$$

where $A_0(\,\cdot\,)$, $B^{\rm o}(\,\cdot\,)$, $C^{\rm o}(\,\cdot\,)$ denote the Fourier symbols of the spatially invariant operators $A^{\rm o}$, $B^{\rm o}$, $C^{\rm o}$, respectively. $\mathcal{A}^{\rm o}_\theta$, $\mathcal{B}^{\rm o}_\theta$, and $\mathcal{C}^{\rm o}_\theta$ can be unbounded operators on ℓ^2 and their diagonal structure follows from the spatial invariance of the operators $A^{\rm o}$, $B^{\rm o}$, and $C^{\rm o}$. In this representation $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ have no particular structure and can be any bounded operators on ℓ^2. To emphasize the notational convention used, we state that the $n^{\rm th}$ row of $\mathcal{E}_\theta$ is $\{\cdots, 0, A_1(\theta_n), 0, A_{-1}(\theta_n), 0, \cdots\}$.

Remark 2: Note that taking $F(x)$ to be a pure cosine is not restrictive. The results obtained here can be easily extended to problems where $F(x)$ is any periodic function with absolutely convergent Fourier series coefficients. ∎

III. Spectral & Stability Analysis

It is shown in [9] that for a general spatially periodic operator A we have

$$\Sigma(A) \;=\; \overline{\bigcup_{\theta \in [0,\Omega)} \Sigma(\mathcal{A}_\theta)}. \qquad (6)$$

In the case where A is spatially invariant (and thus $\mathcal{A}_\theta = \mathrm{diag}\,\{\cdots, A_0(\theta_n), \cdots\}$), (6) further simplifies to

$$\Sigma(A) \;=\; \overline{\bigcup_{k \in \mathbb{R}} \Sigma_{\rm p}\big(A_0(k)\big)}. \qquad (7)$$

Example 1: Let $A = -(\partial_x^2 + \varkappa^2)^2$ with domain

$$\mathscr{D} = \{\phi \in L^2(\mathbb{R}) \mid \phi, \frac{d\phi}{dx}, \frac{d^2\phi}{dx^2}, \frac{d^3\phi}{dx^3} \text{ absolutely continuous}, \frac{d^4\phi}{dx^4} \in L^2(\mathbb{R})\}. \qquad (8)$$

An integration by parts shows that A is a self-adjoint operator and thus closed. $A_0(k) = -(k^2 - \varkappa^2)^2$ is the Fourier symbol of A, see Figure 1 (top). Since $A_0(\,\cdot\,)$ is scalar, $\Sigma_{\rm p}\big(A_0(k)\big) = A_0(k)$ for every k. It is easy to see that $A_0(\,\cdot\,)$ takes every real negative value and thus from (7) A has continuous spectrum $\Sigma(A) = (-\infty, 0]$, see Figure 1 (center). ∎

Remark 3: When A is spatially invariant, a helpful way to think about $\Sigma(A)$ in terms of the symbol $A_0(\,\cdot\,)$ of A is suggested by the previous example. First plot $\Sigma_{\rm p}\big(A_0(\,\cdot\,)\big)$ in the 'complex-plane $\times$ spatial-frequency' space, as in Figure 1 (top) of Example 1. Then the orthogonal projection onto the complex plane of this plot would give $\Sigma(A)$. This can be considered as a geometric interpretation of (7). In Example 1 since $A_0(\,\cdot\,)$ is real scalar and takes only negative values this projection yields only the negative real axis. But in general if $A_0(\,\cdot\,) \in \mathbb{C}^{q\times q}$ this projection would lead to q curves in the complex plane.

Notice also that in this setting $\Sigma(\mathcal{A}_\theta)$ is the projection onto the complex plane of samples of $\Sigma_{\rm p}\big(A_0(\,\cdot\,)\big)$ taken at $k = \theta + \Omega n$, $n \in \mathbb{Z}$, in the 'complex-plane $\times$ spatial-frequency' space. As θ varies in $[0, \Omega)$ these projections trace out $\Sigma(A)$ in the complex plane. This can be considered as a geometric interpretation of (6). Figure 1 (bottom) shows the said samples in the 'complex-plane $\times$ spatial-frequency' space for a scalar A. ∎

We next introduce a special subclass of *holomorphic* (or *analytic*) semigroups. The reader is referred to [4]–[6] for a detailed discussion. Suppose A is densely defined, $\rho(A)$ contains a sector of the complex plane $|\arg(z-\alpha)| \le \frac{\pi}{2} + \gamma$, $\gamma > 0$, $\alpha \in \mathbb{R}$, and there exists some $M > 0$ such that

$$\|(zI - A)^{-1}\| \le \frac{M}{|z - \alpha|} \quad \text{for} \quad |\arg(z-\alpha)| \le \frac{\pi}{2} + \gamma. \qquad (9)$$

Then A generates a holomorphic semigroup and we write $A \in \mathscr{H}(\gamma, \alpha, M)$ [6], [4]. We say that A is *sectorial* if A belongs to some $\mathscr{H}(\gamma, \alpha, M)$.

Finally, a semigroup is called exponentially stable if there exist positive constants M and ϱ such that [7]

$$\|e^{At}\| \le M e^{-\varrho t} \quad \text{for} \quad t \ge 0.$$

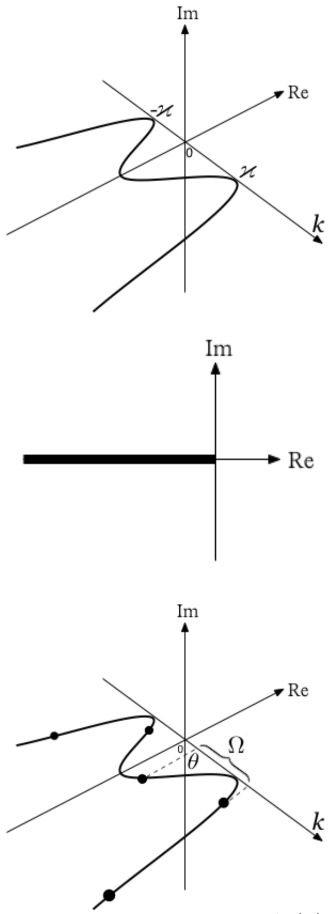

Fig. 1. Top: Representation of the symbol $A_0(\cdot)$ of Example 1 in 'complex-plane $\times$ spatial-frequency' space. Center: $\Sigma(A)$ in the complex plane. Bottom: For spatially invariant A, the (diagonal) elements of $\mathcal{A}_\theta$ are samples of the Fourier symbol $A_0(\cdot)$.

Theorem 1: Assume that A is sectorial. Then if $\Sigma(A) \subset \mathbb{C}^-$, A generates an exponentially stable semigroup.

Proof: If A is sectorial it defines a holomorphic semigroup, and thus e^{At} is differentiable for $t > 0$ [5]. Then [3] shows that this is sufficient for the spectrum-determined growth condition to hold. Since $\Sigma(A) \subset \mathbb{C}^-$ and A is sectorial, then $\Sigma(A)$ is bounded away from the imaginary axis. Let $\omega_\sigma := \sup_{z \in \Sigma(A)} \mathrm{Re}(z)$. Then $\omega_\sigma < 0$ and A generates an exponentially stable semigroup. ■

IV. Stability and the Spectrum-Determined Growth Condition

In the literature on semigroups there exist examples in which $\Sigma(A)$ lies entirely inside $\mathbb{C}^-$ but $\|e^{At}\|$ does not decay exponetially; see [1] and more recently [2]. In such cases it is said that the semigroup does not satisfy the *spectrum-determined growth condition* [3]. The determining factor in the examples presented in [1] and [2] is the accumulation of the eigenvalues of $\mathcal{A}_\theta$ around $\pm j\infty$ in the form of Jordan blocks of ever-increasing size (i.e. as the eigenvalues tend to $\pm j\infty$ their algebraic multiplicity increases while their geometric multiplicity stays equal to one). But such cases are ruled out when one deals with holomorphic semigroups, which is the reason we consider such semigroups in this paper.

Our ultimate aim in this section is to verify exponential stability. By Theorem 1, in order to prove exponential stability of a holomorphic semigroup with infinitesimal generator A, it is sufficient to show that $\Sigma(A) \subset \mathbb{C}^-$. Hence, in the first part of this section we give conditions under which the A operators described by (1) generate holomorphic semigroups. In the second part we find sufficient conditions which guarantee $\Sigma(A) \subset \mathbb{C}^-$.

Once again, the setup is that of (1). In addition, we assume that $A_0(k) \in \mathbb{C}^{q\times q}$ is diagonalizable for every $k \in \mathbb{R}$.

Conditions for Sectorial A

To find conditions under which A in (1) will define a holomorphic semigroup we have to check the condition (9), i.e., we have to verify whether $\|(zI - A)^{-1}\| \le M/|z - \alpha|$ for all z belonging to some sector of $\mathbb{C}$. This involves finding the inverse of the operator $zI - A$ and then calculating its norm. Such a computation can in general be very difficult. On the other hand, finding $\|(zI - A^{\mathrm{o}})^{-1}\|$ is very easy because of the spatial invariance of A^{o}. Indeed, from the norm-preserving property of the Fourier transform it follows that $\|(zI - A^{\mathrm{o}})^{-1}\| = \sup_{k\in\mathbb{R}} \|(zI - A_0(k))^{-1}\|$.

Thus to establish conditions for A to be sectorial we again use perturbation theory. We first find conditions under which A^{o} is sectorial. We then show that $A = A^{\mathrm{o}} + \epsilon\, E$ remains sectorial if E is 'weaker' than A^{o} in a certain sense we will describe, and if ϵ is small enough.

In the next theorem we present a condition for a spatially invariant A^{o} with Fourier symbol $A_0(\cdot)$ to be sectorial.

Theorem 2: Let $A_0(k)$ be diagonalizable for every $k \in \mathbb{R}$, and let $U(k)$ be the transformation that diagonalizes $A_0(k)$, i.e., $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$ with $\Lambda(k)$ diagonal. Let $\kappa(k) := \|U(k)\|\,\|U^{-1}(k)\|$ denote the condition number of $U(k)$. If $\sup_{k\in\mathbb{R}} \kappa(k) < \infty$, and for every $k \in \mathbb{R}$ the resolvent set $\rho(A_0(k))$ contains a sector of the complex plane $|\arg(z - \alpha)| < \frac{\pi}{2} + \gamma$ with $\gamma > 0$ and $\alpha \in \mathbb{R}$ both independent of k, then A^{o} is sectorial.

Proof: See Appendix. ■

This theorem has a particularly simple interpretation when $A_0(\cdot)$ is scalar. In this case $\kappa(k) = 1$ for all $k \in \mathbb{R}$. Now since $A_0(\cdot)$ traces a curve in the complex plane, by Theorem 2 if this curve stays outside some sector $|\arg(z - \alpha)| \le \frac{\pi}{2} + \gamma$, $\gamma > 0$, of the complex plane then A^{o} is sectorial.

The following theorem is the main result of this section and uses the notion of *relative boundedness* of one unbounded operator with respect to another unbounded operator [4] to prove that $A = A^{\mathrm{o}} + \epsilon\, E$ is sectorial.

Theorem 3: Let A^{o} with domain $\mathscr{D}$ be a closed operator, and $A^{\mathrm{o}} \in \mathscr{H}(\gamma, \alpha, M)$. Let $E = B^{\mathrm{o}}\, F\, C^{\mathrm{o}}$ with domain $\mathscr{D}' \supset \mathscr{D}$ be relatively bounded with respect to A^{o} such that

$$\|E\,\psi\| \;\le\; a\,\|\psi\| \;+\; b\,\|A^{\mathrm{o}}\,\psi\|, \quad \psi \in \mathscr{D}, \qquad (10)$$

with $0 \le a < \infty$ and $0 \le b\,|\epsilon| < 1/(1 + M)$. Then $A = A^{\mathrm{o}} + \epsilon\, E$ is a closed and sectorial operator.

Proof: From (10) we have

$$\|\epsilon E\,\psi\| \;\le\; a\,|\epsilon|\,\|\psi\| \;+\; b\,|\epsilon|\,\|A^{\mathrm{o}}\,\psi\|$$

Then from [6, Thm 4.5.7] it follows that $A = A^o + \epsilon E$ is sectorial for $0 \le b\,|\epsilon| < 1/(1+M)$. Also since $M > 0$, then $b\,|\epsilon| < 1$ and [4, Thm IV.1.1] gives that A is closed. ∎

This theorem says that if A^o is sectorial and closed, then so is $A = A^o + \epsilon E$ if E is weaker than A^o in the sense of (10) and if $|\epsilon|$ is small enough. Notice that at this point, condition (10) can not be reduced to a condition in terms of Fourier symbols (i.e. a condition that can be checked pointwise in k) as in Theorem 2. This is because E is not spatially invariant. But once the exact form of the operators B^o and C^o is known, (10) *can* be simplified to a condition on the symbols of A^o, B^o and C^o. Let us clarify this statement with the aid of an example.

Example 2: Consider the periodic PDE

$$\begin{aligned} \partial_t \psi \;&=\; -(\partial_x^2 + \varkappa^2)^2 \psi \;-\; c\,\psi \;+\; \epsilon\,\partial_x \cos(\Omega x)\,\psi \;+\; u \\ y \;&=\; \psi, \end{aligned}$$

where $\psi \in \mathscr{D}$, and $\mathscr{D}$ is defined as in (8). It is easy to see that $A^o = -(\partial_x^2 + \varkappa^2)^2 - c$ is sectorial by Theorem 2 and closed by Example 1, $B^o = \partial_x$, and $C^o = \delta(x)$ [the identity convolution operator]. By formal differentiation we have

$$E\,\psi \;=\; \partial_x \cos(\Omega x)\,\psi \;=\; -\Omega\,\sin(\Omega x)\,\psi + \cos(\Omega x)\,\partial_x\,\psi.$$

Using the triangle inequality and $\|\sin(\Omega x)\| = \|\cos(\Omega x)\| = 1$ we have

$$\|E\,\psi\| \;\le\; |\Omega|\,\|\psi\| \;+\; \|\partial_x\,\psi\|. \tag{11}$$

Thus we have effectively 'commuted out' the bounded spatially periodic component of E, and are left with only spatially invariant operators on the right of (11). Hence, after applying a Fourier transfomation to the right side of (11), a sufficient condition for (10) to hold is that

$$|\Omega| \;+\; |k| \;\le\; a \;+\; b\,|(k^2 - \varkappa^2)^2 + c|, \qquad k \in \mathbb{R},$$

which can be shown to be satisfied for large enough $a > |\Omega|$ and $b > 0$. Using Theorem 3, A is sectorial and closed for small enough $|\epsilon|$. ∎

Remark 4: The above example makes clear the notion of E being 'weaker' than A^o that we mentioned at the beginning of this subsection. If in Example 2 we had $B^o = \partial_x^\nu$ and $C^o = \partial_x^\mu$ and $\nu + \mu = 5$, then E would contain a 5^{th} order derivative, whereas the highest order of ∂_x in A^o is 4. This would mean that (10) can not be satisfied for any choice of a and b. ∎

Conditions for $\Sigma(A) \subset \mathbb{C}^-$

The final step in establishing exponential stability is to show that $\Sigma(A) \subset \mathbb{C}^-$. Unfortunately it is in general very difficult to find the spectrum of an infinite-dimensional operator. Thus we proceed as follows. We consider the (block) diagonal operators $\mathcal{A}_\theta^o$, $\theta \in [0, \Omega)$. This allows us to extend Geršgorin-type methods (existing for finite-dimensional matrices) to find bounds on the location of $\Sigma(\mathcal{A}_\theta)$, $\mathcal{A}_\theta = \mathcal{A}_\theta^o + \epsilon\,\mathcal{E}_\theta$. In turn, we use this to find conditions under which $\Sigma(\mathcal{A}_\theta) \subset \mathbb{C}^-$, and thus $\Sigma(A) \subset \mathbb{C}^-$.

In locating the spectrum of a finite-dimensional matrix $T \in \mathbb{C}^{q\times q}$, the theory of Geršgorin circles [12] provides us with a region of the complex plane that contains the eigenvalues of T. This region is composed of the union of q disks, the centers of which are the diagonal elements of T, and their radii depend on the magnitude of the off-diagonal elements [12]. This theory has also been extended to the case of finite-dimensional block matrices (i.e., matrices whose elements are themselves matrices) in [13]. Next, we further extend this theory to include bi-infinite (block) matrices $\mathcal{A}_\theta$.

For the operator $A = A^o + \epsilon\,E$, take $\mathfrak{B}_k$ to be the set of complex numbers z that satisfy

$$\sigma_{\min}\big(zI - A_0(k)\big) \le\; |\epsilon|\, \Big(\|A_{-1}(k)\| + \|A_1(k)\|\Big), \tag{12}$$

where $\sigma_{\min}\big(zI - A_0(k)\big)$ denotes the smallest singular value of the matrix $zI - A_0(k)$.

Lemma 4: For every θ we have $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta$, where

$$\mathfrak{S}_\theta \;=\; \overline{\bigcup_{n\in\mathbb{Z}} \mathfrak{B}_{\theta_n}}.$$

Proof: See Appendix. ∎

Example 3: Let us consider the periodic PDE [9]

$$\begin{aligned} \partial_t \psi \;&=\; -(\partial_x^2 + \varkappa^2)^2 \psi \;-\; c\,\psi \;+\; \epsilon\,\cos(\Omega x)\,\partial_x\,\psi \;+\; u \\ y \;&=\; \psi, \end{aligned} \tag{13}$$

where $\psi \in \mathscr{D}$, and $\mathscr{D}$ defined as in (8). Comparing (13) and (1) we have

$$A_0(k) = -(k^2 - \varkappa^2)^2 - c, \quad B^o(k) = 1, \quad C^o(k) = jk,$$
$$B(k) = 1, \quad C(k) = 1, \quad L = \frac{1}{2}.$$

From (4)–(5), $A_1(k) = \frac{j}{2}(k - \Omega)$, $A_{-1}(k) = \frac{j}{2}(k + \Omega)$, and thus $\|A_{-1}(k)\| + \|A_1(k)\| = \frac{1}{2}(|k - \Omega| + |k + \Omega|)$. Hence (12) leads to

$$\begin{aligned} \sigma_{\min}\big(zI - A_0(k)\big) = |zI - A_0(k)| &\le \frac{|\epsilon|}{2}(|k - \Omega| + |k + \Omega|) \\ &= \begin{cases} \Omega\,|\epsilon| & |k| \le \Omega \\ |k|\,|\epsilon| & |k| \ge \Omega \end{cases}, \end{aligned}$$

which means that the set $\mathfrak{S}_\theta$ is composed of the union of disks with centers at $A_0(\theta_n)$ and radii $\frac{|\epsilon|}{2}(|\theta_n - \Omega| + |\theta_n + \Omega|)$. Figure 2 (top & center) show $\mathfrak{S}_\theta$ in the complex-plane × spatial-frequency space and in $\mathbb{C}$ respectively.[1] ∎

Remark 5: The set

$$\begin{aligned} \Sigma_\varepsilon(T) &:= \{z \in \mathbb{C} \mid \sigma_{\min}(zI - T) \le \varepsilon\} \\ &= \{z \in \mathbb{C} \mid \|(zI - T)\varphi\| \le \varepsilon \text{ for some } \|\varphi\| = 1\} \\ &= \{z \in \mathbb{C} \mid z \in \Sigma_{\text{p}}(T + Z) \text{ for some } \|Z\| \le \varepsilon\} \end{aligned} \tag{14}$$

is called the *ε-pseudospectrum* of the matrix T [14]. Clearly $\Sigma_{\varepsilon'}(T) \subseteq \Sigma_\varepsilon(T)$ if $\varepsilon' \le \varepsilon$, and $\Sigma_\varepsilon(T) = \Sigma_{\text{p}}(T)$ for $\varepsilon = 0$. The pseudospectrum is composed of small sets around the eigenvalues of T. For instance if T has simple eigenvalues, then for small enough values of ε the pseudospectrum

[1] We would like to point out that Figure 2 (top) is technically incorrect; once the spatially invariant system is perturbed by a spatially periodic perturbation it is no longer spatially invariant and thus can not be fully represented by a Fourier symbol. Hence its spectrum can no longer be demonstrated in the complex-plane × spatial-frequency space.

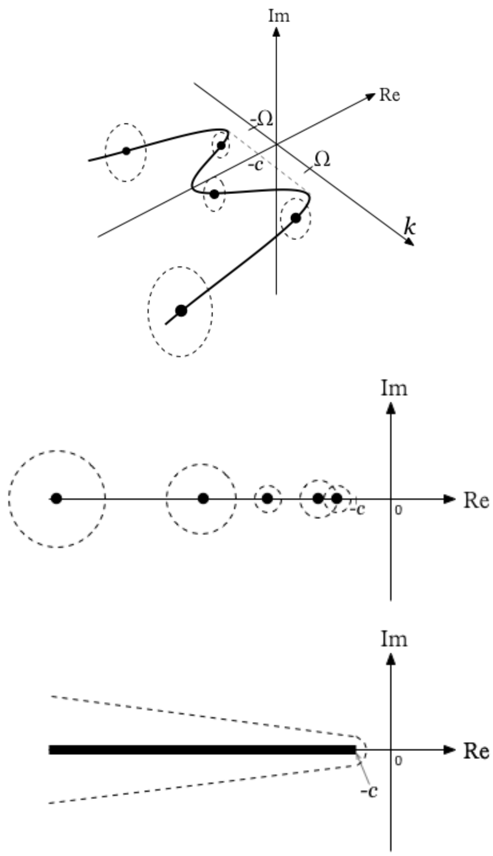

Fig. 2. Top: The $\mathfrak{B}_{\theta_n}$ regions viewed in the 'complex-plane $\times$ spatial-frequency' space (the disks are parallel to the complex plane). Center: $\Sigma(\mathcal{A}_\theta)$ is contained inside the union of the regions $\mathfrak{B}_{\theta_n}$. Bottom: The bold line shows $\Sigma(A^{\rm o})$ and the dotted region contains $\Sigma(A)$, $A = A^{\rm o} + \epsilon E$.

consists of disjoint compact and convex neighborhoods of each eigenvalue [15]. ■

Remark 6: Comparing (14) and the definition of $\mathfrak{B}_k$ in (12), it is clear that $\mathfrak{B}_k = \Sigma_\varepsilon(A_0(k))$ with $\varepsilon = |\epsilon| \left(\|A_{-1}(k)\| + \|A_1(k)\|\right)$. Thus for every $k \in \mathbb{R}$, (12) defines a closed region of $\mathbb{C}$ that includes the eigenvalues of $A_0(k)$. ■

We now employ Lemma 4 to determine whether $\Sigma(A)$ resides completely inside $\mathbb{C}^-$, as needed to assess system stability. Take $\mathfrak{D}_\varepsilon$ to be the closed disk of radius ε and center at the origin, and $\mathfrak{B}_k$ to be the region described by (12). Define the sum of sets by $\mathfrak{U}_1 + \mathfrak{U}_2 = \{z \mid z = z_1 + z_2, z_1 \in \mathfrak{U}_1, z_1 \in \mathfrak{U}_1\}$. Also, for every $k \in \mathbb{R}$ let $\lambda_{\max}(k)$ represent the eigenvalue of $A_0(k)$ with the maximum real part, and let $\kappa(k)$ be defined as in Theorem 2.

Theorem 5: For every k, $\mathfrak{B}_k$ is contained inside $\Sigma_{\rm p}(A_0(k)) + \mathfrak{D}_{r(k)}$ with

$$r(k) \;=\; |\epsilon| \left(\|A_{-1}(k)\| + \|A_1(k)\|\right) \kappa(k).$$

In particular, if $\Sigma(A^{\rm o}) \subset \mathbb{C}^-$ and

$$r(k) \;<\; \left|\mathrm{Re}(\lambda_{\max}(k))\right| + \beta \qquad (15)$$

for every $k \in \mathbb{R}$ and some $\beta < 0$ independent of k, then $\Sigma(A) \subset \mathbb{C}^-$.

Proof: See Appendix. ■

Example 4: Once again we use the scalar system of Example 3. $\kappa(k) = 1$ since $A_0(k)$ is scalar, $|\mathrm{Re}\big(\lambda_{\max}(k)\big)| = |(k^2 - \varkappa^2)^2 + c|$, and

$$\|A_{-1}(k)\| + \|A_1(k)\| \;=\; \frac{1}{2} \left(|k - \Omega| + |k + \Omega|\right).$$

Thus condition (15) becomes

$$\frac{|\epsilon|}{2} \left(|k - \Omega| + |k + \Omega|\right) \;<\; |(k^2 - \varkappa^2)^2 + c| + \beta.$$

If this condition is satisfied for some $\beta < 0$, the dotted region in Figure 2 (bottom) will remain inside $\mathbb{C}^-$ and thus $\Sigma(A) \subset \mathbb{C}^-$. ■

To recap, to assess exponential stability we first find sufficient conditions on A such that it belongs to the class of operators for which the spectrum-determined growth condition holds. These are conditions under which A is sectorial. We then find sufficient conditions for A to have $\mathbb{C}^-$ spectrum. We do this via an extension of Geršgorin circles to bi-infinite (block) matrices.

V. Conclusions and Future Work

In this paper we study the problem of exponential stability for a class of spatially periodic systems. We do this by (i) finding conditions under which the infinitesimal generator is sectorial (i.e., generates a holomorphic semigroup) and thus satisfies the spectrum-determined growth condition, and, (ii) deriving conditions which guarantee that the infinitesimal generator has open LHP spectrum.

Future work in this direction would include extending this procedure to larger classes of systems, for example those in which the Fourier symbol of the spatially invariant part of the infinitesimal generator contains Jordan blocks.

VI. Appendix

Proof of Theorem 2

It is shown in [16] that a sufficient condition for $A^{\rm o}$ to be sectorial is that $\rho(A^{\rm o})$ contain some right half plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq \mu\}$, and

$$\|z(zI - A^{\rm o})^{-1}\| \leq M \quad \text{for} \quad \mathrm{Re}(z) \geq \mu,$$

for some $\mu \geq 0$ and $M \geq 1$.

Now since $A_0(k) \in \mathbb{C}^{q \times q}$ is diagonalizable for every k, there exists a matrix $U(k)$ such that $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$ with $\Lambda(k)$ a diagonal matrix. Let $\lambda_i(k)$, $i = 1, \cdots, q$ denote the diagonal elements of $\Lambda(k)$. Clearly these are also the eigenvalues of $A_0(k)$. Thus we have

$$\begin{aligned}
\|z(zI - A^{\rm o})^{-1}\| \;&\leq\; \sup_{k \in \mathbb{R}} \left(\|z\big(zI - A_0(k)\big)^{-1}\|\right) \\
&\leq\; \sup_{k \in \mathbb{R}} \Big(\|U(k)\|\, \|U^{-1}(k)\| \\
&\qquad\qquad \|z\big(zI - \Lambda(k)\big)^{-1}\|\Big) \\
&=\; \sup_{k \in \mathbb{R}} \left(\kappa(k)\; \frac{|z|}{\mathrm{dist}[z, \Sigma_{\rm p}\big(A_0(k)\big)]}\right) \\
&\leq\; \kappa_{\max} \sup_{k \in \mathbb{R}} \left(\frac{|z|}{\mathrm{dist}[z, \Sigma_{\rm p}\big(A_0(k)\big)]}\right),
\end{aligned}$$

where $\kappa_{\max} := \sup_{k\in\mathbb{R}} \kappa(k)$.

Let us now choose the positive scalar $M' = (1+\kappa_{\max})M$, $M > 1$, and consider for a given k the region of the complex plane where

$$\kappa_{\max}\ \frac{|z|}{\operatorname{dist}[z, \Sigma_{\rm p}(A_0(k))]}\ \geq\ M'.$$

This region (which contains the eigenvalues $\lambda_i(k)$) is contained inside the union of the circles

$$\kappa_{\max}\ \frac{|z|}{|z-\lambda_i(k)|}\ \geq\ M', \qquad i = 1, \cdots, q,$$

which are themselves contained inside the larger circles

$$|z-\lambda_i(k)|\ \leq\ \frac{|\lambda_i(k)|}{M}, \qquad i = 1, \cdots, q. \tag{A1}$$

Notice that (A1) describes circles whose radii increase like $|\lambda_i(k)|/M$, $M > 1$, as their centers $\lambda_i(k)$ become distant from the origin. Clearly a sufficient condition for these circles to belong to some open half plane $\{z \in \mathbb{C}\,|\,\operatorname{Re}(z) < \mu\}$ for all $k \in \mathbb{R}$ and large enough M is that $\Sigma_{\rm p}\big(A_0(k)\big)$, $k \in \mathbb{R}$, reside outside some sector $|\arg(z-\alpha)| \leq \frac{\pi}{2} + \gamma$, $\gamma > 0$, of the complex plane.

Finally, if the circles (A1) are contained in some open half plane $\{z \in \mathbb{C}\,|\,\operatorname{Re}(z) < \mu\}$ for all $k \in \mathbb{R}$, then for $\operatorname{Re}(z) \geq \mu$, $z \in \rho(A_0(k))$ and we have

$$\kappa_{\max}\ \sup_{k\in\mathbb{R}} \Big(\frac{|z|}{\operatorname{dist}[z, \Sigma_{\rm p}\big(A_0(k)\big)]}\Big)\ \leq\ M$$

and thus $\|z(zI - A^{\rm o})^{-1}\| \leq M$ for $\operatorname{Re}(z) \geq \mu$.

Proof of Lemma 4

We use $\Pi_N\, T\, \Pi_N$ to denote the $(2N+1)\times(2N+1)$ [block] truncation of an operator T on ℓ^2, where Π_N is the projection defined by

$$\operatorname{diag}\Big\{\cdots, 0, \underbrace{I, \cdots, \overset{\overset{\textbf{center}}{\downarrow}}{I}, \cdots, I}_{2N+1 \text{ times}}, 0, \cdots\Big\},$$

where I is the identity matrix. Notice that $\Pi_N\, T\, \Pi_N$ is still an operator on ℓ^2; it made from the bi-infinite T by replacing all entries outside the center $(2N+1)\times(2N+1)$ block with zeros. We now form the finite-dimensional matrix $\Pi_N\, \mathcal{A}_\theta\, \Pi_N\big|_{\Pi_N \ell^2}$ by restricting $\Pi_N\, \mathcal{A}_\theta\, \Pi_N$ to the finite-dimensional space $\Pi_N \ell^2$. Clearly $\Pi_N\, \mathcal{A}_\theta\, \Pi_N\big|_{\Pi_N \ell^2}$ has pure point spectrum. Hence using a generalized form of the Geršgorin Circle Theorem [13] for finite-dimensional (block) matrices, we conclude that

$$\Sigma\big(\Pi_N\, \mathcal{A}_\theta\, \Pi_N\big|_{\Pi_N \ell^2}\big)\ \subset\ \bigcup_{|n|\leq N} \mathfrak{B}_{\theta_n}\ \subseteq\ \overline{\bigcup_{n\in\mathbb{Z}} \mathfrak{B}_{\theta_n}}$$

where $\mathfrak{B}_{\theta_n}$ are regions of $\mathbb{C}$ defined by (12). Since this holds for all $N \geq 0$, we have $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta$.

Proof of Theorem 5

If $U(k)$ diagonalizes $A_0(k)$, $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$, and $\kappa(k) = \|U(k)\|\,\|U^{-1}(k)\|$ denotes the condition number of $U(k)$, then from [17] the pseudospectrum of $A_0(k)$ satisfies

$$\Sigma_{\rm p}(A_0(k)) + \mathfrak{D}_\varepsilon\ \subseteq\ \Sigma_\varepsilon(A_0(k))\ \subseteq\ \Sigma_{\rm p}(A_0(k)) + \mathfrak{D}_{\varepsilon\kappa(k)} \tag{A2}$$

for all $\varepsilon \geq 0$. Thus the first statement of the Theorem follows immediately from (A2) and the fact that $\mathfrak{B}_k = \Sigma_\varepsilon(A_0(k))$ with $\varepsilon = |\epsilon|\,\big(\|A_{-1}(k)\| + \|A_1(k)\|\big)$ [see Remark 6]. To prove the second statement, let $\mathbb{C}^-_\beta$ denote all complex numbers with real part less than $\beta \in \mathbb{R}$. It follows from $\Sigma(A^{\rm o}) \subset \mathbb{C}^-$ that $\Sigma(\mathcal{A}^{\rm o}_\theta) \subset \mathbb{C}^-$ for every θ. If (15) holds then

$$\mathfrak{B}_{\theta_n}\ \subseteq\ \Sigma_{\rm p}(A_0(\theta_n)) + \mathfrak{D}_{r(\theta_n)}\ \subset\ \mathbb{C}^-_\beta$$

for every $n \in \mathbb{Z}$, and from Lemma 4 we have $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta = \overline{\bigcup_{n\in\mathbb{Z}} \mathfrak{B}_{\theta_n}} \subset \mathbb{C}^-_{\beta'}$ for some $\beta < \beta' < 0$ and every θ. Thus $\Sigma(A) \subset \mathbb{C}^-$.

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