A Frequency Domain Analysis and Synthesis of the Passivity of Sampled-Data Systems

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***Abstract*— We present a frequency-domain solution to the sampled-data passivity problem. Our analysis is exact in that we take into account the intersample behavior of the sampled-data system. We use Frequency-Response operators and derive a necessary and sufficient condition on the discrete controller that renders a passive closed-loop system. We then find a finite-dimensional system whose closed-loop passivity is equivalent to that of the original one, thus solving the sampled-data passivity synthesis problem. Computations of the equivalent system are carried out in the frequency domain.**

I. Introduction

Sampled-data systems have been the subject of extensive research for over a decade [1], [2], [3], [4], [5], [6]. The two main approaches to this problem have been the state-space [3], and the frequency-domain [7], [5], [6]. We will utilize the frequency-domain framework of [5], [7], [8] here, where the system is represented by Frequency Response (*FR*) operators.

The importance of verifying the passivity of a control system is well understood. One of its major uses is in proving closed-loop stability. An interesting and practical application of this is illustrated in [9] to prove the stability of haptic systems, where the authors use energy methods and physical insight to find necessary and sufficient conditions for the passivity of a sampled-data control system. It is our aim here to provide a general method for the analysis and synthesis of such problems, and show that the results of [9] turn out to be a special case.

In this paper we study the passivity problem of sampled-data systems, assuming SISO systems and controllers for simplicity. We start from the passivity condition on the closed-loop (infinite-dimensional matrix) FR operator, and then derive a necessary and sufficient *scalar* condition on the controller. We also propose an "equivalent" finite-dimensional discrete-time system that displays the same passivity properties as the original system, for a given controller. We name this the *passivity-equivalent* system.

The works most closely related to our's are that of [10], [11]. It is shown in [10] that if the D_{11} matrix of the open-loop system G is zero, then a necessary condition for the passivity of the sample-data system is that G_{11} have relative degree one. Our work also verifies this result using different methodology. [11] proposes a solution to the sample-data passivity problem in the state-space setting by checking whether the $\mathcal{H}_\infty$-norm of the Cayley transform of the sampled-data system is less than or equal to unity. In this paper, we find a direct frequency domain approach to this problem.

Our presentation is organized as follows: In Section II we develop the mathematical tools needed to represent sampled-data systems as FR operators in Section III. Then Section IV proceeds to find a scalar check, in terms of the controller, for the passivity of the closed-loop system. Section V deals with finding the finite-dimensional passivity-equivalent system. Finally, in Section VI we show that our passivity-equivalent system is actually rational. The problem of the passivity of haptic systems is then solved using our method in Section VII and well-known existing results reproduced.

Notation: Notation is fairly standard; we use capital letters for systems, and small letters for signals. We use the same notation for signals/systems and their Fourier transforms, but the distinction will be clear from the context. P^* is the adjoint of the operator P, which for matrices is equivalent to the complex-conjugate transpose of the matrix. We also denote the complex-conjugate of a complex scalar by *, for consistency.

This work was supported by the Air Force Office of Scientific Research under Grant FA9550-04-1-0207 and the National Science Foundation under Grant ECS-0323814.

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II. Preliminaries

In this section we will derive the frequency-domain representation of linear time-periodic systems. We follow [8] with changes as needed for the study of temporal (causal) systems.

A. The Lifting Operator $\mathcal{W}_T$

For every $f \in L^2[0, \infty)$, the lifted function can be defined as

$$\hat{f}_n(\hat{t}) := f(nT + \hat{t}) = \mathcal{W}_T f, \quad n = 0, 1, 2, ..., \quad \hat{t} \in [0, T].$$

Clearly $\{\hat{f}_n(\hat{t})\} \in l^2_{L^2[0,\, T]}$, i.e., for any given $n = 0, 1, 2, ..., \ \hat{f}_n(\hat{t}) \in L^2[0,\, T]$.

The lifting operator, $\mathcal{W}_T$, is one-to-one and onto and has a well-defined inverse $\mathcal{W}_T^{-1}$

$$f = \mathcal{W}_T^{-1} \hat{g}, \quad f(t) = \hat{g}_n(t - nT),$$
$$n = 0, 1, 2, ..., \quad nT \le t \le (n+1)T.$$

The action of the operator $\mathcal{W}_T^{-1}$ is exactly the opposite of the one caused by the lifting operator. Namely, $\mathcal{W}_T^{-1}$ pastes together a sequence of functions, each in $L^2[0,\, T]$, thus giving a function $f \in L^2[0,\, \infty)$, [3].

B. The $\mathcal{Z}$-transform

The $\mathcal{Z}$-transform of the sequence $\{\hat{f}_n(\hat{t})\}$ evaluated on the unit circle, $z = e^{j\theta}$, is determined by

$$\hat{f}_\theta(\hat{t}) := \sum_{n=0}^{\infty} \hat{f}_n(\hat{t}) e^{-j\theta n}.$$

$\{\hat{f}_\theta(\hat{t})\} \in L^2_{L^2[0,\,T]}[0,\, 2\pi]$, i.e. for any given $\theta \in [0,\, 2\pi]$, $\hat{f}_\theta(\hat{t}) \in L^2[0,\, T]$.

C. The $\mathcal{V}_\theta$-transform

Define the following θ-parameterized functions in $L^2[0,\, T]$

$$\vartheta_{\theta,k}(\hat{t}) := \frac{1}{\sqrt{T}} e^{j\frac{2\pi k + \theta}{T}\hat{t}}, \qquad k \in \mathbb{Z}.$$

It can be shown [7] that for any θ, $\{\vartheta_{\theta,k}(\hat{t})\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2[0,\, T]$, and so $\hat{f}_\theta(\hat{t}) \in L^2[0,\, T]$ has representation

$$\hat{f}_\theta(\hat{t}) = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}} \hat{f}_\theta(k) \vartheta_{\theta,k}(\hat{t}) \tag{1}$$

for some sequence $\{\hat{f}_\theta(k)\} \in l^2$. Let us now find $\{\hat{f}_\theta(k)\}$. It is easy to prove that if $f(t) \in L^2[0, \infty)$ has the Fourier transform $f(j\omega)$

$$f(j\omega) := \int_0^\infty f(t) e^{-j\omega t} dt,$$

then $\hat{f}_\theta(\hat{t})$ can be written as

$$\hat{f}_\theta(\hat{t}) = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}} f(j\frac{2\pi k + \theta}{T}) \vartheta_{\theta,k}(\hat{t}). \tag{2}$$

Comparison of (1) and (2) gives $\hat{f}_\theta(k) = f(j\frac{2\pi k + \theta}{T})$, and we will often switch between these two notations.

It is convenient to write $\{\hat{f}_\theta(k)\}$ as a vector in l^2, and then (1) becomes

$$\hat{f}_\theta(\hat{t}) = \frac{1}{\sqrt{T}} \begin{bmatrix} \cdots & \vartheta_{\theta,k}(\hat{t}) I & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{f}_\theta(k) \\ \vdots \end{bmatrix},$$

where I is the identity matrix, with dimension equal to the Euclidean dimension of f. Define $\mathcal{V}_\theta^{-1} := \frac{1}{\sqrt{T}} \begin{bmatrix} \cdots & \vartheta_{\theta,k}(\hat{t}) I & \cdots \end{bmatrix}$. Clearly $\mathcal{V}_\theta$ can be viewed as a unitary transformation from $L^2_{L^2[0,\,T]}[0,\, 2\pi]$ to $L^2_{l^2}[0,\, 2\pi]$, and

$$\mathcal{V}_\theta \hat{f}_\theta(\hat{t}) =: \begin{bmatrix} \vdots \\ \hat{f}_\theta(k) \\ \vdots \end{bmatrix},$$

which we denote by $\mathcal{V}_\theta \hat{f}_\theta(\hat{t}) = f_\theta$, f_θ representing the l^2 vector of $\{\hat{f}_\theta(k)\}$.

D. Application to Signals and Systems

Define the transformation $\mathcal{M} := \mathcal{V}_\theta \mathcal{Z} \mathcal{W}_T$, application of which gives the FR operator representation. Let us now demonstrate the affect of $\mathcal{M}$.

Consider the LTI system $y(t) = Gu(t)$ with frequency-domain representation $y(j\omega) = G(j\omega)u(j\omega)$. Now take $\omega = \frac{2k\pi + \theta}{T}$, where $k \in \mathbb{Z}$ and $\theta \in [0, 2\pi]$,

$$y(j\frac{2k\pi + \theta}{T}) = G(j\frac{2k\pi + \theta}{T}) u(j\frac{2k\pi + \theta}{T}),$$

which we denote, as before, by $\hat{y}_\theta(k) = \hat{G}_\theta(k)\hat{u}_\theta(k)$. Now stacking the $\hat{u}_\theta(k)$ and $\hat{y}_\theta(k)$ in vectors, one can write the above system in matrix form [1]

$$\begin{bmatrix} \vdots \\ \hat{y}_\theta(k) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & G(j\frac{2k\pi + \theta}{T}) & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{u}_\theta(k) \\ \vdots \end{bmatrix},$$

which we denote $y_\theta = G_\theta u_\theta$. Notice that, by what we have shown so far, $u_\theta = \mathcal{M}u(t)$ and $y_\theta = \mathcal{M}y(t)$, and thus $\mathcal{M}G\mathcal{M}^{-1} = G_\theta = \mathrm{diag}\{G(j\frac{2k\pi + \theta}{T})\}$. In other words, the FR matrix representation of an LTI system G is $G_\theta = \mathrm{diag}\{G(j\frac{2k\pi + \theta}{T})\}$, $\theta \in [0, 2\pi]$.

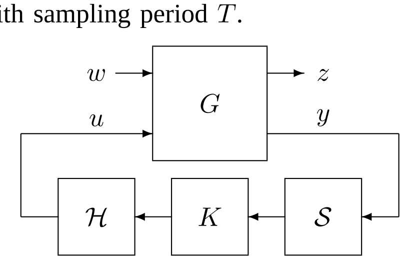
It is also easy to show that the ideal sampler $\mathcal{S}$ and zero-order hold $\mathcal{H}$ operators in sampled-data systems have the FR matrix representation

$$\mathcal{S}_\theta = \begin{bmatrix} \cdots I \cdots \end{bmatrix}, \qquad \mathcal{H}_\theta = \frac{1 - e^{-j\theta}}{T} \begin{bmatrix} \vdots \\ \dfrac{I}{j\frac{2k\pi + \theta}{T}} \\ \vdots \end{bmatrix},$$

where T is the sampling period.

III. Frequency Domain Analysis of Sampled-Data Systems

Consider the general internally stable feedback connection of the continuous-time system G, $\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix}$ and the discrete-time controller K, $u_n = Ky_n$, through the sample and hold operators $\mathcal{S}$ and $\mathcal{H}$ with sampling period T.



[1] In the present text, the elements of a matrix that are left blank, are zero.

The equations governing the system are

$$\begin{bmatrix} z(t) \\ y_n \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12}\mathcal{H} \\ \mathcal{S}G_{21} & \mathcal{S}G_{22}\mathcal{H} \end{bmatrix} \begin{bmatrix} w(t) \\ u_n \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathcal{S} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathcal{H} \end{bmatrix} \begin{bmatrix} w(t) \\ u_n \end{bmatrix}.$$

Applying the operator $\begin{bmatrix} \mathcal{W}_T & 0 \\ 0 & I \end{bmatrix}$ on both sides and simplifying, we get

$$\begin{bmatrix} \hat{z}_n(\hat{t}) \\ y_n \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \hat{\mathcal{S}} \end{bmatrix} \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \hat{\mathcal{H}} \end{bmatrix} \begin{bmatrix} \hat{w}_n(\hat{t}) \\ u_n \end{bmatrix},$$

where $\hat{G}_{ij} := \mathcal{W}_T G_{ij} \mathcal{W}_T^{-1}$, $\hat{\mathcal{S}} := \mathcal{S}\mathcal{W}_T^{-1}$, and $\hat{\mathcal{H}} := \mathcal{W}_T\mathcal{H}$. Notice that all the operators in the above equation have now a shift-invariant structure. Hence one can apply the $\mathcal{Z}$-transform to them. Taking $\mathcal{Z}\big|_{z=e^{j\theta}}$-transform, $\begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \mathcal{Z} \end{bmatrix}$, of both sides, we arrive at

$$\begin{bmatrix} \hat{z}_\theta(\hat{t}) \\ y_\theta \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \hat{\mathcal{S}}_\theta \end{bmatrix} \begin{bmatrix} \hat{G}_{11\theta} & \hat{G}_{12\theta} \\ \hat{G}_{21\theta} & \hat{G}_{22\theta} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \hat{\mathcal{H}}_\theta \end{bmatrix} \begin{bmatrix} \hat{w}_\theta(\hat{t}) \\ u_\theta \end{bmatrix}$$

with $[.]_\theta := \mathcal{Z}[.]\mathcal{Z}^{-1}$ for all systems above. Finally, applying $\begin{bmatrix} \mathcal{V}_\theta & 0 \\ 0 & I \end{bmatrix}$ to both sides, we have

$$\begin{bmatrix} \hat{z}_\theta(k) \\ y_\theta \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \hat{\mathcal{S}}_\theta(k) \end{bmatrix} \begin{bmatrix} \hat{G}_{11\theta}(k) & \hat{G}_{12\theta}(k) \\ \hat{G}_{21\theta}(k) & \hat{G}_{22\theta}(k) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \hat{\mathcal{H}}_\theta(k) \end{bmatrix} \begin{bmatrix} \hat{w}_\theta(k) \\ u_\theta \end{bmatrix}$$

where $\hat{G}_{ij\theta}(k) := \mathcal{V}_\theta \hat{G}_{ij\theta} \mathcal{V}_\theta^{-1}$, $\hat{\mathcal{S}}_\theta(k) := \hat{\mathcal{S}}_\theta \mathcal{V}_\theta^{-1}$, and $\hat{\mathcal{H}}_\theta(k) := \mathcal{V}_\theta \hat{\mathcal{H}}_\theta$. Now if we use the l^2-vector representation for $\hat{z}_\theta(k)$ and $\hat{w}_\theta(k)$, namely z_θ and w_θ, and define $G_{ij\theta} := \mathrm{diag}\{G_{ij}(j\frac{2k\pi+\theta}{T})\}$, we obtain

$$\begin{bmatrix} z_\theta \\ y_\theta \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathcal{S}_\theta \end{bmatrix} \begin{bmatrix} G_{11\theta} & G_{12\theta} \\ G_{21\theta} & G_{22\theta} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathcal{H}_\theta \end{bmatrix} \begin{bmatrix} w_\theta \\ u_\theta \end{bmatrix} = \begin{bmatrix} G_{11\theta} & G_{12\theta}\mathcal{H}_\theta \\ \mathcal{S}_\theta G_{21\theta} & \mathcal{S}_\theta G_{22\theta}\mathcal{H}_\theta \end{bmatrix} \begin{bmatrix} w_\theta \\ u_\theta \end{bmatrix} =: \begin{bmatrix} \widetilde{G}_{11\theta} & \widetilde{G}_{12\theta} \\ \widetilde{G}_{21\theta} & \widetilde{G}_{22\theta} \end{bmatrix} \begin{bmatrix} w_\theta \\ u_\theta \end{bmatrix}.$$

where

$$\widetilde{G}_{11\theta} = \begin{bmatrix} \ddots & & \\ & G_{11}(j\frac{2k\pi+\theta}{T}) & \\ & & \ddots \end{bmatrix},$$

$$\widetilde{G}_{12\theta} = \frac{1-e^{-j\theta}}{T} \begin{bmatrix} \vdots \\ G_{12}(j\frac{2k\pi+\theta}{T})\frac{I}{j\frac{2k\pi+\theta}{T}} \\ \vdots \end{bmatrix},$$

$$\widetilde{G}_{21\theta} = \begin{bmatrix} \cdots & G_{21}(j\frac{2k\pi+\theta}{T}) & \cdots \end{bmatrix},$$

$$\widetilde{G}_{22\theta} = \sum_{k=-\infty}^{\infty} \frac{1-e^{-j\theta}}{j2k\pi+j\theta} G_{22}(j\frac{2k\pi+\theta}{T}), \quad (3)$$

Observe that $\widetilde{G}_{22\theta} = \mathcal{S}_\theta G_{22\theta}\mathcal{H}_\theta$ above gives exactly the classical formula relating the frequency response of a continuous-time system and its step-invariant discretization [12].

The system $\widetilde{G}_\theta$, together with the controller $u_\theta = K(e^{j\theta})y_\theta$ give the closed-loop operator

$$\widetilde{G}_{zw\theta} = \widetilde{G}_{11\theta} + \widetilde{G}_{12\theta}\,K(e^{j\theta}) \Big(I - \widetilde{G}_{22\theta}\,K(e^{j\theta})\Big)^{-1} \widetilde{G}_{21\theta}. \quad (4)$$

IV. Passivity Analysis

Consider the question of the passivity of the internally stable closed-loop system (4), which amounts to checking the inequality

$$\widetilde{G}_{zw\theta} + \widetilde{G}^*_{zw\theta} > 0 \quad \forall\, \theta \in [0, 2\pi]. \quad (5)$$

Assume that the controller $K(e^{j\theta})$, and define $L_\theta := K(e^{j\theta})\big(1 - \widetilde{G}_{22\theta}\;K(e^{j\theta})\big)^{-1}$. Then $G_{zw\theta} = \widetilde{G}_{11\theta} + \widetilde{G}_{12\theta}L_\theta\widetilde{G}_{21\theta}$, and (5) becomes

$$\begin{aligned} \widetilde{G}_{zw\theta} + \widetilde{G}^*_{zw\theta} &= (\widetilde{G}_{11\theta} + \widetilde{G}_{12\theta}L_\theta\widetilde{G}_{21\theta}) \\ &\quad + (\widetilde{G}_{11\theta} + \widetilde{G}_{12\theta}L_\theta\widetilde{G}_{21\theta})^* \\ &= \widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta} + \widetilde{G}_{12\theta}L_\theta\widetilde{G}_{21\theta} \\ &\quad + \widetilde{G}^*_{21\theta}L^*_\theta\widetilde{G}^*_{12\theta} > 0. \end{aligned} \quad (6)$$

For mere notational clarity, let us define the bi-infinite (diagonal) matrix $\Lambda := \widetilde{G}_{11\theta}$, and the two bi-infinite vectors $\zeta := \widetilde{G}_{12\theta}$, and $\xi := (L_\theta\widetilde{G}_{21\theta})^*$, for a given value of θ. Equation (6) now reads

$$\Lambda + \Lambda^* + \zeta\xi^* + \xi\zeta^* > 0. \quad (7)$$

Essentially (7) requires checking the positivity of a bi-infinite matrix for every $\theta \in [0, 2\pi]$. Yet an observation simplifies the problem considerably; one can rewrite (7)

$$\Lambda + \Lambda^* + \frac{1}{2}(\zeta+\xi)(\zeta+\xi)^* - \frac{1}{2}(\zeta-\xi)(\zeta-\xi)^* > 0,$$

or simply

$$\Lambda + \Lambda^* + \phi\phi^* - \psi\psi^* > 0, \quad (8)$$

where $\phi := \frac{1}{\sqrt{2}}(\zeta+\xi)$ and $\psi := \frac{1}{\sqrt{2}}(\zeta-\xi)$.

Remark 1: In (8), $\Lambda+\Lambda^*$ depends on G_{11} only, while the two one-dimensional (i.e. with one-dimensional range) operators $\phi\phi^*$ and $\psi\psi^*$ contain the controller. In other words, the closed-loop system with a discrete-time controller constitutes a two-dimensional perturbation of the open-loop system G_{11}.

Lemma 1: A necessary condition for (8) is that $\Lambda+\Lambda^*$ have at most one nonpositive eigenvalue.

Proof: Let us denote the (real) eigenvalues of a self-adjoint matrix A by $\lambda_i(A)$, $i = 1, 2, \cdots$, where $\lambda_1 \le \lambda_2 \le \cdots$. Assume now that $\Lambda + \Lambda^*$ has two nonpositive eigenvalues, i.e., $\lambda_1(\Lambda + \Lambda^*) \le \lambda_2(\Lambda + \Lambda^*) \le 0$. Then it can be shown (Corollary 4.3.3 of [13]) that $\lambda_i(\Lambda+\Lambda^*-\psi\psi^*) \le \lambda_i(\Lambda+\Lambda^*)$, $i = 1, 2, \cdots$, which means, in particular, that $\Lambda+\Lambda^*-\psi\psi^*$ will have at least two nonpositive eigenvalues. Now using the fact that the eigenvalues of $\Lambda+\Lambda^*-\psi\psi^*$ and $\Lambda+\Lambda^*+\phi\phi^*-\psi\psi^*$ interlace (Theorem 4.3.3 of [13]) we have, in particular, that $\lambda_1(\Lambda+\Lambda^*+\phi\phi^*-\psi\psi^*)\leq\lambda_2(\Lambda+\Lambda^*-\psi\psi^*)\leq 0$. Hence $\Lambda+\Lambda^*+\phi\phi^*-\psi\psi^*$ is not positive. [2] ■

Remark 2: Notice the requirements this places on satisfying the passivity condition. Since $\Lambda+\Lambda^* = \text{diag}\{\big(G_{11}+G_{11}^*\big)(j\frac{2k\pi+\theta}{T})\}$, if for some $\theta=\theta_-$, $\big(G_{11}+G_{11}^*\big)(j\frac{2k\pi+\theta_-}{T})\leq 0$ for more that one $k\in\mathbb{Z}$, then $\Lambda+\Lambda^*$ will have more than one nonpositive eigenvalue, and (8) can not be satisfied, by Lemma 1.

Theorem 2: For a given sampling period T, a necessary condition for (8) is that $\big(G_{11}+G_{11}^*\big)(j.)$ not be nonpositive on any connected set Ω of support greater than π/T.

Proof: It is easy to show that in such a case $\Lambda+\Lambda^*$ will have more than one nonpositive eigenvalue for some θ, where we are also using the fact that $\big(G_{11}+G_{11}^*\big)(-j\omega)=\big(G_{11}+G_{11}^*\big)(j\omega)$ for real systems. The details are omitted for brevity. ■

Corollary 3: Some immediate consequences of the above theorem are that for the passivity of the closed-loop system, we need $D_{11}+D_{11}^*\geq 0$, and also, if $D_{11}=0$, then $G_{11}(s)$ can not have relative degree greater than one (otherwise $\big(G_{11}+G_{11}^*\big)(j\omega)\leq 0$ on an unbounded set $\omega\in[\omega_0,\infty)$ for some ω_0). This verifies the results of [10].

In this paper, we will henceforth assume that the open-loop system G_{11} is passive, i.e., $\big(G_{11}+G_{11}^*\big)(j\omega)>0\quad\forall\omega\in\mathbb{R}$. This means that $\Lambda+\Lambda^*>0$, and using Corollary 4.3.3 of [13], that $\Lambda+\Lambda^*+\phi\phi^*>0$. Thus applying Schur complements, we can formally write the following set of equivalent inequalities

$$\Lambda+\Lambda^*+\phi\phi^*-\psi\psi^*>0$$
$$\Updownarrow$$
$$\begin{bmatrix}\Lambda+\Lambda^*+\phi\phi^* & \psi\\ \psi^* & 1\end{bmatrix}>0$$
$$\Updownarrow$$
$$1-\psi^*(\Lambda+\Lambda^*+\phi\phi^*)^{-1}\psi>0. \tag{9}$$

We have effectively transformed the question of the positivity of a bi-infinite matrix, to that of a scalar. This, however, comes at a price; the inverse of the bi-infinite matrix $\Lambda+\Lambda^*+\phi\phi^*$ has to be found. But for this we can use the matrix inversion lemma

$$(\Lambda+\Lambda^*+\phi\phi^*)^{-1}=(\Lambda+\Lambda^*)^{-1}-\\ (\Lambda+\Lambda^*)^{-1}\phi\big(1+\phi^*(\Lambda+\Lambda^*)^{-1}\phi\big)^{-1}\phi^*(\Lambda+\Lambda^*)^{-1},$$

[2]To be precise, we have to mention that the results used from [13] are stated there for finite-dimensional matrices only. But one can always take finite-dimensional projections (i.e. finite truncations) of the above infinite-dimensional matrices, and show that our conclusions hold for all such projections, and then pass to the limit.

to get

$$1-\psi^*(\Lambda+\Lambda^*+\phi\phi^*)^{-1}\psi=1-\psi^*(\Lambda+\Lambda^*)^{-1}\psi\\ +\frac{\big(\psi^*(\Lambda+\Lambda^*)^{-1}\phi\big)\big(\phi^*(\Lambda+\Lambda^*)^{-1}\psi\big)}{1+\phi^*(\Lambda+\Lambda^*)^{-1}\phi}.$$

So checking (9) has now boiled down to checking

$$1+\frac{|\psi^*(\Lambda+\Lambda^*)^{-1}\phi|^2}{1+\phi^*(\Lambda+\Lambda^*)^{-1}\phi}>\psi^*(\Lambda+\Lambda^*)^{-1}\psi. \tag{10}$$

Now since $\Lambda+\Lambda^*>0$, we can define a new inner product on a subspace of l^2

$$\langle\rho,\eta\rangle_\Lambda:=\rho^*(\Lambda+\Lambda^*)^{-1}\eta,$$

which induces a new norm on l^2, $||\eta||_\Lambda^2:=\langle\eta,\eta\rangle_\Lambda$. Substituting $\phi=(\zeta+\xi)/\sqrt{2}$ and $\psi=(\zeta-\xi)/\sqrt{2}$ back into (10) and simplifying, we finally arrive at

$$|1+\langle\zeta,\xi\rangle_\Lambda|^2>||\zeta||_\Lambda^2||\xi||_\Lambda^2,$$

or, equivalently,

$$\Big|1+\langle\widetilde{G}_{12\theta},\widetilde{G}_{21\theta}^*L_\theta^*\rangle_\Lambda\Big|^2>||\widetilde{G}_{12\theta}||_\Lambda^2||\widetilde{G}_{21\theta}^*L_\theta^*||_\Lambda^2. \tag{11}$$

Replacing $L_\theta:=K(e^{j\theta})\big(1-\widetilde{G}_{22\theta}\,K(e^{j\theta})\big)^{-1}$, one can simplify (11) even further to get

$$\Big|K(e^{j\theta})^{-1}-\big(\widetilde{G}_{22\theta}-\langle\widetilde{G}_{12\theta},\widetilde{G}_{21\theta}^*\rangle_\Lambda\big)\Big|^2>\\ ||\widetilde{G}_{12\theta}||_\Lambda^2||\widetilde{G}_{21\theta}^*||_\Lambda^2. \tag{12}$$

Notice that the evaluation of terms of the form $\langle\ .\ ,\ .\ \rangle_\Lambda$ and $||\ .\ ||_\Lambda^2$ involves the calculation of infinite sums. We will show how to evaluate these sums in Section VI.

Remark 3: If D_{11} is zero, i.e., G_{11} has no direct feedthrough term, then $\Lambda+\Lambda^*$ in (10) is a compact operator and hence $(\Lambda+\Lambda^*)^{-1}$ will be an unbounded operator. This puts certain constraints on the admissible ϕ and ψ so that each term in (10) is bounded. This, in turn, translates to conditions on G_{12} and G_{21} such that $\langle\ .\ ,\ .\ \rangle_\Lambda$ and $||\ .\ ||_\Lambda^2$ in (12) remain bounded. It can be shown that a necessary and sufficient condition for this would be that $G_{12}(s)$ and $G_{21}(s)$ be strictly proper.

We have thus proved

Theorem 4: Assuming passivity of the open-loop system G_{11}, an internally stable closed-loop sampled-data system is passive if and only if $K(e^{j\theta})^{-1}$ lies outside a circle with center $\widetilde{G}_{22\theta}-\langle\widetilde{G}_{12\theta},\widetilde{G}_{21\theta}^*\rangle_\Lambda$ and radius $||\widetilde{G}_{12\theta}||_\Lambda\,||\widetilde{G}_{21\theta}^*||_\Lambda$ for all $\theta\in[0,2\pi]$.

V. The Passivity-Equivalent System

Since each of the terms in (12) is a scalar function of θ, one could ask the question whether it is possible to find *finite-dimensional* functions $\bar{G}_{11\theta}$, $\bar{G}_{12\theta}$, and $\bar{G}_{21\theta}$, such that

$$\Big|K^{-1}-\big(\widetilde{G}_{22\theta}-\langle\widetilde{G}_{12\theta},\widetilde{G}_{21\theta}^*\rangle_\Lambda\big)\Big|^2>||\widetilde{G}_{12\theta}||_\Lambda^2||\widetilde{G}_{21\theta}^*||_\Lambda^2\\ \Updownarrow \tag{13}\\ \Big|K^{-1}-\big(\bar{G}_{22\theta}-\langle\bar{G}_{12\theta},\bar{G}_{21\theta}^*\rangle\big)\Big|^2>||\bar{G}_{12\theta}||^2||\bar{G}_{21\theta}^*||^2$$

where in the latter inequality, the inner product and norm are defined on a finite-dimensional Euclidean space according to $\langle \rho, \eta \rangle := \rho^*(\bar{G}_{11\theta} + \bar{G}^*_{11\theta})^{-1}\eta$, assuming, of course, that a $\bar{G}_{11\theta}$ can be found such that $\bar{G}_{11\theta} + \bar{G}^*_{11\theta} > 0$. [3]

Clearly, a sufficient condition for (13) would be

$$\bar{G}^*_{12\theta}(\bar{G}_{11\theta} + \bar{G}^*_{11\theta})^{-1}\bar{G}_{12\theta} = \widetilde{G}^*_{12\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}_{12\theta}, \quad (14)$$

$$\bar{G}_{21\theta}(\bar{G}_{11\theta} + \bar{G}^*_{11\theta})^{-1}\bar{G}^*_{21\theta} = \widetilde{G}_{21\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}^*_{21\theta}, \quad (15)$$

$$\bar{G}_{21\theta}(\bar{G}_{11\theta} + \bar{G}^*_{11\theta})^{-1}\bar{G}_{12\theta} = \widetilde{G}_{21\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}_{12\theta}. \quad (16)$$

For the sake of clarity, we will define new notation and write $\bar{G}_{11\theta}$, $\bar{G}_{12\theta}$, and $\bar{G}_{21\theta}$, as $\bar{G}_{11\theta} =: \breve{G}_{11\theta}I$, $\bar{G}_{12\theta} =: U_{12}\breve{G}_{12\theta}$ and $\bar{G}_{21\theta} =: \breve{G}_{21\theta}U^*_{21}$ respectively, where $\breve{G}_{11\theta}$, $\breve{G}_{12\theta}$ and $\breve{G}_{21\theta}$ are now scalar functions, I is the identity operator, and U_{12}, U_{21} are left-inner operators (i.e. $U^*_{12}U_{12} = 1$, $U^*_{21}U_{21} = 1$), all with appropriate input-output dimensions (i.e., $\bar{G}_{11\theta}$, $\bar{G}_{12\theta}$, and $\bar{G}_{21\theta}$ can form an admissible system). Our aim now is to find scalar $\breve{G}_{ij\theta}$, and U_{ij} with minimum input-output dimensions, such that (14)-(16) are satisfied.

Treating $\breve{G}_{11\theta}$ as a parameter for the time being, one can determine $\breve{G}_{12\theta}$ from (14) [4]

$$\breve{G}^*_{12\theta}\breve{G}_{12\theta} = \frac{\widetilde{G}^*_{12\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}_{12\theta}}{(\breve{G}_{11\theta} + \breve{G}^*_{11\theta})^{-1}}, \quad (17)$$

and $\breve{G}_{21\theta}$ from (15)

$$\breve{G}_{21\theta}\breve{G}^*_{21\theta} = \frac{\widetilde{G}_{21\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}^*_{21\theta}}{(\breve{G}_{11\theta} + \breve{G}^*_{11\theta})^{-1}}. \quad (18)$$

Notice that $\breve{G}_{12\theta}$ and $\breve{G}_{21\theta}$ can always be found from these equations by performing a spectral factorization on the positive operators on the right hand side. This leaves (16) yet to be satisfied, which can be simplified to

$$U^*_{21}U_{12} = \frac{\widetilde{G}_{21\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}_{12\theta}}{\breve{G}_{21\theta}(\breve{G}_{11\theta} + \breve{G}^*_{11\theta})^{-1}\breve{G}_{12\theta}}. \quad (19)$$

It is easy to show that the right side of (19) is a complex scalar of absolute value less than or equal to one. As a result, U_{12} and U_{21} can not both be (unitary) scalars. Hence we will consider higher dimensional left-inner operators, namely $U_{12}, U_{21} \in \mathbb{C}^2$.

[3] Notice that internal stability is preserved here, because $K(e^{j\theta})$ and $\widetilde{G}_{22\theta} = \bar{G}_{22\theta}$, being scalar functions of θ, remain the same in both the original infinite-dimensional system and its finite-dimensional equivalent.

[4] In writing equations (17) and (18), we use the fact that $\breve{G}_{11\theta}$, being a scalar function, commutes with $\breve{G}_{12\theta}$ and $\breve{G}_{21\theta}$, and that

$$(U_{12}\breve{G}_{12\theta})^*(U_{12}\breve{G}_{12\theta}) = \breve{G}^*_{12\theta}U^*_{12}U_{12}\breve{G}_{12\theta} = \breve{G}^*_{12\theta}\breve{G}_{12\theta},$$
$$(\breve{G}_{21\theta}U^*_{21})(\breve{G}_{21\theta}U^*_{21})^* = \breve{G}_{21\theta}U^*_{21}U_{21}\breve{G}^*_{21\theta} = \breve{G}_{21\theta}\breve{G}^*_{21\theta}.$$

We propose $\bar{G}_{11\theta} = \breve{G}_{11\theta}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\bar{G}_{12\theta} = U_{12}\breve{G}_{12\theta} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\breve{G}_{12\theta}$, $|\alpha|^2 + |\beta|^2 = 1$, and $\bar{G}_{21\theta} = \breve{G}_{21\theta}U^*_{21} = \breve{G}_{21\theta}\begin{bmatrix} 1 & 0 \end{bmatrix}$. Choose $\breve{G}_{12\theta}$, $\breve{G}_{21\theta}$, and $\breve{G}_{11\theta}$ as to satisfy (17), (18), and insure (19) by

$$\begin{aligned} U^*_{21}U_{12} &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \\ &= \frac{\widetilde{G}_{21\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}_{12\theta}}{\breve{G}_{21\theta}(\breve{G}_{11\theta} + \breve{G}^*_{11\theta})^{-1}\breve{G}_{12\theta}}. \end{aligned}$$

α is thus found, and so is β from $|\beta|^2 = 1 - |\alpha|^2$.

Remark 4: Interestingly, one can see that $\breve{G}_{11\theta}$ can always be chosen independently of $\breve{G}_{12\theta}$, $\breve{G}_{21\theta}$, U_{12}, and U_{21}, as long as $\breve{G}_{11\theta} + \breve{G}^*_{11\theta} > 0$. As a matter of fact, in (17)-(19) one can "absorb" the dynamics of $\breve{G}_{11\theta}$ into that of $\breve{G}_{12\theta}$, $\breve{G}_{21\theta}$, U_{12}, and U_{21}. So for example, one possible choice for $\breve{G}_{11\theta}$ could be $\breve{G}_{11\theta} + \breve{G}^*_{11\theta} = 1$, i.e., $\breve{G}_{11\theta} = 1/2$.

We have thus proved

Theorem 5: The closed-loop connection of $\begin{bmatrix} G_{11}(j\omega) & G_{12}(j\omega) \\ G_{21}(j\omega) & G_{22}(j\omega) \end{bmatrix}$ and the discrete-time scalar controller $K(e^{j\theta})$ is passive if and only if the closed-loop connection of $\begin{bmatrix} \bar{G}_{11\theta} & \bar{G}_{12\theta} \\ \bar{G}_{21\theta} & \bar{G}_{22\theta} \end{bmatrix}$ and $K(e^{j\theta})$ is passive, where

$$\bar{G}_{11\theta} = \breve{G}_{11\theta}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{G}_{12\theta} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\breve{G}_{12\theta},$$
$$\bar{G}_{21\theta} = \breve{G}_{21\theta}\begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \bar{G}_{22\theta} = \widetilde{G}_{22\theta},$$

with $\breve{G}_{11\theta}$, $\breve{G}_{12\theta}$, $\breve{G}_{21\theta}$, α, and β scalar functions previously derived.

VI. Realization as Discrete-Time Systems with Rational Transfer Functions

In this section we aim to show that the scalar discrete-time systems $\breve{G}_{11\theta}$, $\breve{G}_{21\theta}$, $\breve{G}_{12\theta}$, α, and β, and hence $\bar{G}_{11\theta}$, $\bar{G}_{12\theta}$, and $\bar{G}_{21\theta}$ obtained in the previous section, can actually be found as *rational* transfer functions.

Let us take $\breve{G}_{11\theta} = 1/2$ (see Remark 4). We next prove that $\breve{G}_{21\theta}$ is rational. For notational brevity, let us introduce $\omega_k := \dfrac{2k\pi + \theta}{T}$. From (3) and (18),

$$\widetilde{G}_{21\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}^*_{21\theta} = \sum_{k=-\infty}^{\infty} G_{21}(j\omega_k)\big(G_{11}(j\omega_k) + G^*_{11}(j\omega_k)\big)^{-1}G^*_{21}(j\omega_k). \quad (20)$$

Hagiwara *et al* [5] give a method for the calculation of infinite sums of the form $\sum_{k=-\infty}^{\infty} G(j\dfrac{2k\pi + \theta}{T})$, using the impulse modulation formula. Namely, for a general system $G(s)$ with state-space description (A, B, C),

$$\sum_{k=-\infty}^{\infty} G(j\frac{2k\pi + \theta}{T}) = e^{j\theta}C(e^{j\theta} - e^{AT})^{-1}B.$$

Assuming convergence of (20), (see Remark 3) define $h(e^{j\theta}) = \dfrac{v(e^{j\theta})}{u(e^{j\theta})} := \widetilde{G}_{21\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}^*_{21\theta} \geq 0$, where v and u are co-prime polynomials in $e^{j\theta}$. Then $|u(e^{j\theta})|^2 h(e^{j\theta}) \geq 0$ is a polynomial in $e^{j\theta}$, and we can apply

Theorem 6: (Fejér-Riesz [14]) Any trigonometric polynomial $f(e^{j\theta}) = \sum_{-n}^{n} a_k e^{kj\theta}$ that is nonnegative on the unit circle Γ, has the form $f(e^{j\theta}) = |g(e^{j\theta})|^2$, where $g(e^{j\theta}) = \sum_0^n b_k e^{kj\theta}$ is an analytic trigonometric polynomial such that $g(z) = \sum_0^n b_k z^k$ has no zeros on the unit disk D,

to conclude that there exists a trigonometric polynomial w such that $|u(e^{j\theta})|^2 h(e^{j\theta}) = |w(e^{j\theta})|^2$, which means

$$h(e^{j\theta}) = \frac{|w(e^{j\theta})|^2}{|u(e^{j\theta})|^2} = \frac{ww^*}{\mu\mu^*},$$

where both w and μ have their zeros *inside* the unit disk.

$$\breve{G}_{21\theta}\breve{G}^*_{21\theta} = (\frac{w}{\mu})(\frac{w}{\mu})^* \implies \breve{G}_{21\theta} = \frac{w(e^{j\theta})}{\mu(e^{j\theta})}.$$

Essentially the same procedure is applied to find $\breve{G}_{12\theta}$, using the fact that the right side of equation (17) constitutes a nonnegative rational function.

Now recall that $\alpha = \dfrac{\widetilde{G}_{21\theta}(\widetilde{G}_{11\theta} + \widetilde{G}^*_{11\theta})^{-1}\widetilde{G}_{12\theta}}{\breve{G}_{21\theta}(\breve{G}_{11\theta} + \breve{G}^*_{11\theta})^{-1}\breve{G}_{12\theta}}$ is a rational function of norm less than or equal to one. The only remaining question is whether we can find rational β such that $|\beta|^2 = 1 - |\alpha|^2$. Consider $|\beta|^2 = 1 - |\alpha|^2 \geq 0$, and assume $\alpha = \dfrac{n}{d}$.

$$1 - (\frac{n}{d})^*(\frac{n}{d}) = \frac{d^*d - n^*n}{d^*d} \geq 0 \implies d^*d - n^*n \geq 0.$$

Since $d^*d - n^*n$ is a nonnegative trigonometric polynomial, we apply the Fejér-Riesz theorem to conclude that there exists a trigonometric polynomial m such that $d^*d - n^*n = |m|^2$, which means

$$|\beta|^2 = \frac{d^*d - n^*n}{d^*d} = \frac{|m|^2}{|d|^2} = \frac{m^*m}{\delta^*\delta},$$

where both m and δ have their zeros *inside* the unit disk.

$$\beta^*\beta = (\frac{m}{\delta})^*(\frac{m}{\delta}) \implies \beta = \frac{m}{\delta}.$$

Hence both U_{12} and U_{21} are known left-inner rational functions.

VII. Passivity of Haptic Systems

Let us apply the results of the previous sections to find necessary and sufficient conditions for the passivity of the haptic system [9]

$$G_{11}(s) = \frac{1}{ms+b}, \qquad G_{12}(s) = -\frac{1}{ms+b},$$
$$G_{21}(s) = \frac{1}{s(ms+b)}, \quad G_{22}(s) = -\frac{1}{s(ms+b)},$$

with $b > 0$ the damping coefficient, and $m > 0$ the mass of the haptic system. A quick calculation shows that

$$\widetilde{G}_{22\theta} - \langle \widetilde{G}_{12\theta}, \widetilde{G}^*_{21\theta} \rangle_\Lambda = \frac{1 - e^{-j\theta}}{2b}\frac{T}{4\sin^2\theta/2},$$
$$||\widetilde{G}_{12\theta}||^2_\Lambda ||\widetilde{G}^*_{21\theta}||^2_\Lambda = \frac{|1 - e^{-j\theta}|^2}{b^2}\frac{T^2}{64\sin^2\theta/2}.$$

Using $r(e^{j\theta}) := -(1 - e^{-j\theta})\dfrac{T}{4\sin^2\theta/2}$ and plugging into (12), we arrive, after simplifications, at $\left|\dfrac{r(e^{j\theta})K(e^{j\theta})}{2b + r(e^{j\theta})K(e^{j\theta})}\right| < 1,\ \theta \in [0, 2\pi]$. This matches precisely the passivity condition obtained in [9].

VIII. Conclusions and Future Work

For the sampled-data system with SISO controller $K(e^{j\theta})$, we give a necessary and sufficient condition on the discrete-time controller such that the closed-loop system is passive. We also propose a two-dimensional discrete-time system that is passivity-equivalent to the original (infinite-dimensional) one, in that, if a given controller K makes the equivalent system passive, it will make the original system passive, and vice versa.

Future work in this direction would include finding a state-space method of deriving the passivity-equivalent system parameters.

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