J. Math. Anal. Appl. 368 (2010) 144–156

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stability in the almost everywhere sense: A linear transfer operator approach

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ARTICLE INFO

Article history:
Received 20 April 2009
Available online 20 February 2010
Submitted by H. Zwart

Keywords:
Almost everywhere stability
Advection equation
Density function

ABSTRACT

The problem of *almost everywhere* stability of a nonlinear autonomous ordinary differential equation is studied using a linear transfer operator framework. The infinitesimal generator of a linear transfer operator (Perron–Frobenius) is used to provide stability conditions of an autonomous ordinary differential equation. It is shown that almost everywhere uniform stability of a nonlinear differential equation, is equivalent to the existence of a non-negative solution for a steady state advection type linear partial differential equation. We refer to this non-negative solution, verifying almost everywhere global stability, as *Lyapunov density*. A numerical method using finite element techniques is used for the computation of Lyapunov density.

1. Introduction

Stability analysis of an ordinary differential equation is one of the most fundamental problems in the theory of dynamical systems. The goal of stability analysis is to obtain an easily verifiable condition for the stability of differential equations. Lyapunov function based methods form the cornerstone of current stability analysis and control design for nonlinear systems [1]. In [2], it was recognized that the Lyapunov function admits a dual counterpart, the so-called "*density function*". The dual is no longer based on energy like criteria provided by Lyapunov function but is based on density propagation and can be used to verify almost everywhere global convergence. The co-design problem of jointly obtaining density function and the controller enjoys the remarkable convexity property. This convexity property is exploited in the work of [3] for the design of stabilizing feedback controllers. Results on the use of density function for verifying almost everywhere stability of stochastic dynamical systems and systems with control inputs also exist in [4,5]. Similarly, converse theorems for almost everywhere stability using density function can be found in [6,7].

In [8], almost everywhere stability problem for discrete time dynamical systems is studied using linear transfer operators, in particular Koopman and Perron–Frobenius (P–F) operators. The Lyapunov measure is introduced as a new tool to verify the weaker notion of almost everywhere stability of an attractor set in nonlinear systems. The Lyapunov measure is shown to be dual to the Lyapunov function. Applications of the Lyapunov measure to the problem of stabilizing control design, optimal stabilization, and for the solution of motion planning problem are studied in [9–11]. The important point being, all the above design problems are posed and solved as a linear programming problems. Set oriented numerical methods developed in [12,13] were used to provide the finite-dimensional approximation of the Lyapunov measure and the controller

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.02.032

in the design problems. One of the motivations of this paper is to introduce the use of a linear partial differential equation (the advection equation) for the problem of stability of nonlinear ordinary differential equations. This will allow us to make use of numerically efficient methods developed for solving partial differential equations for the stability of ordinary differential equations.

In this paper, we continue our investigation on the stability of dynamical systems to develop the continuous time counterpart of the discrete time results published in [8,14]. The infinitesimal generator of the Perron–Frobenius transfer operator is used to prove a notion of almost everywhere stability of an attractor set in continuous time dynamical systems that is stronger than the one discussed in [2]. The main result of this paper shows that almost everywhere uniform stability of an attractor set for a continuous time dynamical system is equivalent to the existence of positive solution to an advection type partial differential equation. We refer to this positive solution as "*Lyapunov density*". This advection type partial differential operator forms the infinitesimal generator for the P–F semigroup. The Lyapunov density introduced in this paper can also be thought of as the density corresponding to the Lyapunov measure introduced in [8].

The advantage of studying the almost everywhere stability problem using the proposed approach in this paper is that the computational methods developed for solving linear partial differential equations can be used to obtain the Lyapunov density. We provide preliminary results on the computation of the Lyapunov density based on the finite element method in lower dimensional dynamical systems. Recent developments in sparse collocation-based numerical methods for the solution of partial differential equation [15,16] provide a promising future direction for the computation of Lyapunov density in higher-dimensional dynamical systems.

The organization of the paper is as follows. In Section 2, we introduce some preliminaries from transfer operator and semigroup theory. In Section 3, we prove the main result of this paper on the use of Lyapunov density for verifying almost everywhere stability of a dynamical system governed by a nonlinear autonomous differential equation. In addition, we also show (Corollary 16) the equivalence between almost everywhere uniform stability with respect to Lebesgue and any other finite measure that has a non-negative integrable density associated with it. Simulation results for the computation of Lyapunov density using finite element methods are presented in Section 4, followed by the conclusion in Section 5.

2. Preliminaries

In this paper, we are interested in the global stability property of an attractor set for the following ordinary differential equation

$$\dot{x} = f(x), \quad x \in X, \tag{1}$$

where f is assumed to be infinitely differentiable and $X \subset \mathbb{R}^n$ is a compact phase space. Our primary motivation for considering a compact phase space is from the point of view of computation. However, for the case where the state space is $\mathbb{R}^n$, we assume that there exists a compact subset $X \subset \mathbb{R}^n$ that is positively invariant for (1) and the question of stability still remains due to the possibility of existence of multiple attractor sets inside X. The theory presented in the paper is also applicable for cases where X has a partial or no boundary. We use the notation $\phi_t(x)$ to denote the solution or *flow* map of (1) at time t, having started from the initial condition x. Eq. (1) can be used to study the evolution of a single trajectory. The evolution of ensembles of trajectories or the densities on the phase space can be studied using a linear operator called the Perron–Frobenius (P–F) operator $\mathbb{P}_t : L^1(X) \to L^1(X)$ which satisfies the following conservation property:

$$\int_A \mathbb{P}_t \rho(x)\, dx = \int_{\phi_{-t}(A)} \rho(x)\, dx = \int_A \rho\big(\phi_{-t}(x)\big) \left| \frac{\partial \phi_{-t}(x)}{\partial x} \right| dx \tag{2}$$

for every measurable set $A \subset X$. Hence, the following identity is true

$$\mathbb{P}_t \rho(x) = \rho\big(\phi_{-t}(x)\big) \left| \frac{\partial \phi_{-t}(x)}{\partial x} \right|, \tag{3}$$

where $|\frac{\partial \phi_{-t}(x)}{\partial x}|$ is the determinant of the Jacobian of the flow map ϕ_{-t}.

Remark 1. We remark that the solutions of (1) given by $\phi_t(x)$ are uniquely maximally defined inside X over an open interval of time (not necessarily $(-\infty, \infty)$). This is the case for trajectories inside X (depending on initial conditions) extend to outside of X backwards in time. For such trajectories, in order to make sense of the integral in (2), we assume that $\rho(x)$ has support entirely on X and is zero outside of X, thereby making $\rho(\phi_{-t}(x)) = 0$ zero for large t in such cases.

Furthermore, the Perron–Frobenius operator introduced above is the semigroup corresponding to the operator $\mathbb{A}\rho = - \nabla \cdot (\rho f)$. In other words, $\mathbb{P}_t \rho_0 = e^{\mathbb{A}t} \rho_0(x)$ describes the evolution of densities ρ via the *advection equation*

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho f) =: \mathbb{A}\rho, \qquad \rho(x, 0) = \rho_0(x). \tag{4}$$

If $(X, \mathcal{B}, \mu)$ is a measure space and $\mathbb{P}_t$ is the Perron–Frobenius operator corresponding to the dynamical system (1), then $\mathbb{P}_t$ satisfies the following properties [17]:

(1) $\mathbb{P}_t(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathbb{P}_t f_1 + \alpha_2 \mathbb{P}_t f_2$ for all $f_1, f_2 \in L^1(X)$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.
(2) $\mathbb{P}_t f \geqslant 0$ if $f \geqslant 0$.
(3) $\int_X \mathbb{P}_t f(x)\,\mu(dx) = \int_X f(x)\,\mu(dx)$.

Roughly speaking, the Perron–Frobenius operator and the advection equation can be thought of as describing the evolution of the density of a fluid as it moves under the influence of the vector-field (1). For more details on the Perron–Frobenius operator, its adjoint and its infinitesimal generator see [17]. We are interested in the global *almost everywhere stability* of an attractor set Λ. We use the following definition for an attractor set [18]. To define an attractor set, we first need the definition of an ω-limit set:

Definition 2 *(ω-Limit set).* A point $x_0 \in X$ is said to be an ω-limit point for a point $x \in X$ if there exists a sequence of time instants $t_k \to \infty$ such that $\phi_{t_k}(x) \to x_0$ as $k \to \infty$. The set of all ω-limit points $\omega(x)$ for x is called the ω-limit set $\omega(x)$.

The definition of an attractor set is as follows:

Definition 3 *(Attractor set).* A closed set $\Lambda \subset X$ is said to be an invariant set for (1) if for any $x \in \Lambda$, $\phi_t(x) \in \Lambda$ for all $t \in \mathbb{R}$. An invariant set Λ is said to be an attractor set if there exists a neighborhood $V \supset \Lambda$ such that $\omega(x) \subset \Lambda$ for all point $x \in V$, and the neighborhood V is forward invariant i.e., $\phi_t(x) \in V$ for all $t \geqslant 0$ and for all $x \in V$.

There are two main reasons for studying the almost everywhere global stability of an attractor set as opposed to an invariant set. First, the stability certificate in the form of Lyapunov density that we introduce in this paper diverges to infinity as it approaches the invariant set Λ and hence the density is well defined only outside the neighborhood of an invariant set. Secondly, the local stability assumption of the invariant set allows us to impose appropriate boundary conditions in the solution of partial differential equation. We will discuss in detail about the boundary conditions in Section 4 which deals with numerical simulation. To define almost everywhere stability of an attractor set Λ, we let

$$A_t := \{x \in \Lambda^c\colon\ \phi_t(x) \in A\}$$

for any measurable set $A \subset X \setminus B_\delta$, where $B_\delta \supset \Lambda$ and $B_\delta \subset V$ is a δ neighborhood of the set Λ for some fixed $\delta > 0$.

We next recall the definition of almost everywhere stability as discussed in [8] and the continuous time counterpart of the discrete time version of almost everywhere uniform stability from [14].

Definition 4 *(Almost everywhere stability).* An attractor set Λ is said to be almost everywhere stable with respect to a finite measure m on Λ^c if $m\{x \in \Lambda^c\colon\ \omega(x) \not\subset \Lambda\} = 0$.

Definition 5 *(Almost everywhere uniform stability).* The attractor set $\Lambda \subset B_\delta$ for the differential equation (1) is said to be almost everywhere uniformly stable with respect to a finite measure m on Λ^c if for any given $\epsilon > 0$, there exists a $T(\epsilon) > 0$ such that

$$\int_T^\infty m(A_t)\,dt < \epsilon \tag{5}$$

for all measurable $A \subset X \setminus B_\delta$.

Remark 6. The measure m in the definition of almost everywhere uniform stability will be assumed to be the Lebesgue measure on Λ^c or any measure that is absolutely continuous with respect to Lebesgue measure on Λ^c.

Our next lemma proves that Definition 5 is a stronger notion of stability than Definition 4.

Lemma 7. *An attractor set $\Lambda \subset X$ is almost everywhere stable if it is almost everywhere uniformly stable.*

Proof. The proof is by contradiction. We define the set $S = \{x \in \Lambda^c\colon\ \omega(x) \notin \Lambda\}$. Assume that $m(S) > 0$. It is clear that $S_t = S$, $\forall t > 0$, and $S \subset X \setminus B_\delta$ since B_δ is in the region of attraction of Λ. Hence, by definition of almost everywhere uniform stability, we have $\int_0^\infty m(S_t)\,dt = \int_0^\infty m(S)\,dt < \infty$. This is possible only when $m(S) = m(S_t) = 0$, which contradicts the initial assumption that $m(S) > 0$. This proves the lemma. □

The notion of almost everywhere uniform global stability is strictly stronger than almost everywhere stability of attractors Λ which have a neighborhood V such that $\omega(V) \subset \Lambda$ (see Example 19, Section 3).

For the case when $\Lambda = \{0\}$ is an equilibrium point the almost everywhere stability definition can be stated as follows

$$m\Big\{x \in \Lambda^c\colon \lim_{t\to\infty} \phi_t(x) \neq 0\Big\} = 0.$$

Since almost everywhere global stability is with respect to the set of points which are outside the δ neighborhood of the attractor set, this motivates us to look at the restriction of the Perron–Frobenius semigroup to the space $L^1(X \setminus \Lambda)$. Hence, we define the new semigroup corresponding to the restriction of the flow $\phi_t : \Lambda^c \to \Lambda^c$ as follows

$$\mathbb{P}_t^1 \rho(x) := \rho\big(\phi_{-t}(x)\big)\left|\frac{\partial \phi_{-t}(x)}{\partial x}\right|, \tag{6}$$

where $\rho(x)$ is supported on the set Λ^c. Since Λ is assumed to be invariant for the dynamics defined by (1), so is Λ^c and hence (6) defines a semigroup on Λ^c.

Remark 8. The set B_δ in Definition 5 essentially allows us to talk about evolution of densities that are integrable in steady state and avoids singularities near the invariant set Λ due to accumulation of mass. Hence we assume that the initial density will be supported on $\Lambda^c = X \setminus \Lambda$ and focus on this set to define almost everywhere uniform stability of the attractor set Λ.

We have the following

$$\mathbb{P}_t^1 = \Sigma \mathbb{P}_t : L^1(X \setminus \Lambda) \to L^1(X \setminus \Lambda),$$

where $\Sigma : L^1(X) \to L^1(X \setminus \Lambda)$ is the projection operator, and

$$(\Sigma \rho)(x) = \chi_{X\setminus\Lambda}(x)\rho(x).$$

Let $\mathbb{A}^1$ be the infinitesimal generator corresponding to the semigroup of the restriction $\mathbb{P}_t^1$. The domain of the generators $\mathbb{A}$ and $\mathbb{A}^1$ are denoted by $\mathcal{D}(\mathbb{A})$ and $\mathcal{D}(\mathbb{A}^1)$ respectively and are defined below

$$\begin{aligned} \mathcal{D}(\mathbb{A}) &= \big\{\rho \in W^{1,1}(X)\colon\ \rho|_{\Gamma_i} = 0\big\}, \\ \mathcal{D}\big(\mathbb{A}^1\big) &= \big\{\rho \in W^{1,1}\big(\Lambda^c\big)\colon\ \rho|_{\Gamma_i} = 0\big\}, \end{aligned} \tag{7}$$

where the inflow portion of the boundary of X (if it exists) is denoted by Γ_i and is given as follows

$$\Gamma_i = \big\{x \in \partial X\colon\ f(x)\cdot \vec{\eta}(x) < 0\big\}, \tag{8}$$

where $\vec{\eta}(x)$ is the unit outward normal at the boundary point x and ∂X denotes the set of all boundary points of X. For sets X that don't have a boundary, the homogeneous boundary conditions can be omitted from the domain definitions in (7).

The notation $W^{1,1}(X)$ refers to the space of elements in $L^1(X)$ (which are also distributions) whose first distributional derivative belongs to $L^1(X)$. We assume that the movement of mass inside the compact set X is entirely due to the initial mass density in its interior and there is no influx of mass through the inflow boundary Γ_i. Since the flux in our case is given by $(f(x)\cdot\vec{\eta}(x))\rho(x)$, and on the inflow boundary we have $f(x)\cdot\vec{\eta}(x) < 0$, this means homogeneous Dirichlet conditions i.e. $\rho|_{\Gamma_i} = 0$. We also note that the inflow boundary Γ_i will be the portion of $\Gamma = \partial X$, that is away from Λ due to the attractor property of Λ that is assumed. Hence we have that $\mathcal{D}(\mathbb{A}^1) \subset \mathcal{D}(\mathbb{A})$.

The relation between the infinitesimal generator corresponding to $\mathbb{P}_t^1$ and that of $\mathbb{P}_t$ is established in the following lemma, the proof of which is omitted.

Lemma 9. *Let $\mathbb{A}^1$ and $\mathbb{A}$ be the infinitesimal generators corresponding to the semigroups $\mathbb{P}_t^1$ and $\mathbb{P}_t$ respectively and $\Sigma : L^1(X) \to L^1(X \setminus \Lambda)$ be the projection operator, then we have*

$$\mathbb{A}^1 = \Sigma\mathbb{A} \quad \text{on } L^1(X \setminus \Lambda).$$

3. Main result

In this section, we prove the main result of this paper giving a necessary and sufficient condition for almost everywhere uniform stability of the attractor set Λ. We first state the following lemma establishing the connection between the almost everywhere stability of the attractor set and the asymptotic property of the restricted semigroup $\mathbb{P}_t^1$, the proof of which follows from Definition 5.

Lemma 10. *The attractor set $\Lambda \subset X$ for the system of differential equations* (1) *is almost everywhere uniformly stable with respect to measure m with density ρ_0 if and only if for every $\epsilon > 0$ there exists a $T(\epsilon)$ such that*

$$\int_A \int_T^\infty \mathbb{P}_t^1 \rho_0(x)\, dt\, dx < \epsilon \tag{9}$$

for $A \subset X \setminus B_\delta$, such that $m(A) > 0$.

Next, we have the following important formula, the proof of which follows from standard ODE techniques.

Lemma 11. *Let $\phi_t(x)$ denote the solution of Eq.* (1)*. Then we have the following identity*

$$\det \frac{d\phi_t(x)}{dx} = e^{\int_0^t \nabla \cdot f(\phi_s(x))\, ds}. \tag{10}$$

We now provide the definition of Lyapunov density.

Definition 12 *(Lyapunov density).* We define Lyapunov density with respect to the underlying measure m, which is assumed to be absolutely continuous with respect to the Lebesgue measure on Λ^c with density $\rho_0(x) \in \mathcal{D}(\mathbb{A}^1) \cap L^1(\Lambda^c)$. Lyapunov density with respect to density ρ_0 is defined as any non-negative function $\rho(x) \in \mathcal{D}(\mathbb{A}^1) \cap L^1(X \setminus B_\delta)$ and satisfying the following inequality

$$\mathbb{A}^1 \rho(x) \leqslant -\rho_0(x). \tag{11}$$

Theorem 13. *Let $X \subset \mathbb{R}^n$ be compact and Γ_i defined below denote the inflow part of the boundary ∂X which is assumed to be $\mathcal{C}^2$:*

$$\Gamma_i = \{x \in \partial X\colon\ \vec{f}(x) \cdot \vec{\eta}(x) < 0\}. \tag{12}$$

Then, the attractor set Λ is almost everywhere uniformly stable with respect to the measure m with density $0 \leqslant \rho_0 \in \mathcal{D}(\mathbb{A}^1) \cap L^1(\Lambda^c)$ if and only if there exists a Lyapunov density with respect to density ρ_0.

Proof. To prove the necessity of (11), we construct a solution $\rho(x)$ as follows

$$\rho(x) = \int_0^\infty \mathbb{P}_t^1 \rho_0(x)\, dt. \tag{13}$$

The definition of a.e. uniform stability w.r.t. ρ_0 given by (9) guarantees the convergence of (13) for almost every $x \in X \setminus B_\delta$ with respect to measure m. Furthermore, the definition also guarantees that $\rho(x) \in L^1(X \setminus B_\delta)$. Also, from (3) and (10), we have that $\rho(x) \geqslant 0$. It remains to verify that (13) defines a solution for (11). To see this, we note that $\rho_N(x) = \int_0^N \mathbb{P}_t^1 \rho_0(x)\, dt \in \mathcal{D}(\mathbb{A}^1) \cap L^1(\Lambda^c)$ since $\rho_0 \in \mathcal{D}(\mathbb{A}^1) \cap L^1(\Lambda^c)$. We apply the operator $\mathbb{A}^1$ to ρ_N first. We get the following

$$\mathbb{A}^1 \rho_N(x) = \int_0^N \mathbb{A}^1 \mathbb{P}_t^1 \rho_0(x)\, dt = \int_0^N \frac{d}{dt} \mathbb{P}_t^1 \rho_0(x) = \mathbb{P}_N^1 \rho_0(x) - \rho_0(x).$$

The right-hand side of the last step above along with Definition 5 implies that $\lim_{N\to\infty} \mathbb{A}^1 \rho_N$ exists. This along with the closedness of the operator $\mathbb{A}^1$ (guaranteed by the Hille–Yosida semigroup generation theorem) gives us the following

$$\int_0^\infty \mathbb{A}^1 \mathbb{P}_t^1 \rho_0(x)\, dt = \int_0^\infty \frac{d}{dt} \mathbb{P}_t^1 \rho_0(x)\, dt = \lim_{t\to\infty} \mathbb{P}_t^1 \rho_0(x) - \rho_0(x) = -\rho_0(x),$$

where we have used the semigroup property of $\mathbb{P}_t^1$ to obtain the first equality above and $\lim_{t\to\infty} \mathbb{P}_t^1 \rho_0(x) = 0$ (implied by the definition of a.e. uniform stability) along with $\lim_{t\to 0} \mathbb{P}_t^1 \rho_0(x) = \rho_0(x)$ to obtain the last equality above. This proves the necessity.

To prove the sufficiency of (11), we first find a representation formula for the solution $\rho \in \mathcal{D}(\mathbb{A}^1) \cap L^1(X \setminus B_\delta)$ of (11), which can be rewritten as follows

$$\nabla \cdot (f\rho) \geqslant \rho_0. \tag{14}$$

All calculations from now on are true in the weak sense i.e. in the sense of distributions. Eq. (14) can be further rewritten as

$$\sum_{i=1}^{n} f_i(x)\rho_{x_i} + \rho(x)(\nabla \cdot f) \geqslant \rho_0(x). \tag{15}$$

The characteristic curves for (15) are given by the solution of the following ODE

$$\dot{x}(t) = f\big(x(t)\big), \qquad x(0) = x_0 \in X. \tag{16}$$

Let $\phi_t(x)$ denote the solution of (16). Then (15) can be rewritten as

$$\frac{d}{dt}\rho\big(\phi_t(x)\big) + \rho\big(\phi_t(x)\big)\big(\nabla \cdot f\big(\phi_t(x)\big)\big) \geqslant \rho_0\big(\phi_t(x)\big), \tag{17}$$

which is a first order differential inequality in the t variable. We use Gronwall's inequality to solve (17). We multiply (17) by $e^{\int_0^t \nabla\cdot f(\phi_s(x))\,ds}$ and obtain the following

$$\frac{d}{dt}\big(\rho\big(\phi_t(x)\big)e^{\int_0^t \nabla\cdot f(\phi_s(x))\,ds}\big) \geqslant e^{\int_0^t \nabla\cdot f(\phi_s(x))\,ds}\rho_0\big(\phi_t(x)\big). \tag{18}$$

Hence, we obtain the following solution formula for (14) along the characteristic curves given by the solution of (16):

$$\rho\big(\phi_t(x)\big) \geqslant e^{-\int_0^t \nabla\cdot f(\phi_s(x))\,ds}\rho\big(\phi_0(x)\big) + e^{-\int_0^t \nabla\cdot f(\phi_s(x))\,ds}\int_0^t e^{\int_0^s \nabla\cdot f(x(\tau))\,d\tau}\rho_0\big(\phi_s(x)\big)\,ds. \tag{19}$$

From Lemma 11 we have

$$\det\frac{d(\phi_t(x))}{dx} = e^{\int_0^t \nabla\cdot f(\phi_s(x))\,ds}.$$

Using the above equation and rearranging Eq. (19), we have the following

$$\begin{aligned}
&\left|\frac{d(\phi_t(x))}{dx}\right|\rho\big(\phi_t(x)\big) \geqslant \rho\big(\phi_0(x)\big) + \int_0^t \left|\frac{d(\phi_s(x))}{dx}\right|\rho_0\big(\phi_s(x)\big)\,ds \\
&\quad\Rightarrow\quad \mathbb{P}_{-t}\rho(x) \geqslant \rho\big(\phi_0(x)\big) + \int_0^t \mathbb{P}_{-s}\rho_0(x)\,dt \\
&\quad\Rightarrow\quad \rho(x) \geqslant \mathbb{P}_t\rho(x) + \int_0^t \mathbb{P}_{t-s}\rho_0(x)\,dt.
\end{aligned} \tag{20}$$

Next, we note that $\phi_0(x) = x$ and integrate Eq. (20) in space with respect to $A \subset X \setminus B_\delta$ to obtain the following

$$\begin{aligned}
&\int_A \rho(x)\,dx \geqslant \int_A \mathbb{P}_t\rho(x) + \int_0^t\int_A \mathbb{P}_{t-s}\rho_0(x) \\
&\quad\Rightarrow\quad \int_A \rho(x)\,dx \geqslant \int_A \mathbb{P}_t^1\rho(x) + \int_0^t\int_A \mathbb{P}_{t-s}^1\rho_0(x)\,dx\,ds \\
&\quad\Rightarrow\quad \int_A \rho(x)\,dx \geqslant \int_A \mathbb{P}_t^1\rho(x) + \int_0^t\int_A \mathbb{P}_\tau^1\rho_0(x)\,d\tau.
\end{aligned}$$

Since $\rho(x) \in \mathcal{D}(\mathbb{A}^1) \cap L^1(X \setminus B_\delta)$ and $\rho_0(x) \in \mathcal{D}(\mathbb{A}^1) \cap L^1(\Lambda^c)$ are both non-negative densities, we have the following

$$\int_0^t\int_A \mathbb{P}_\tau^1\rho_0(x)\,d\tau < \int_A \mathbb{P}_t^1\rho(x) + \int_0^t\int_A \mathbb{P}_\tau^1\rho(x)\,d\tau \leqslant \int_A \rho(x)\,dx < \infty, \quad \forall t > 0, \tag{21}$$

which is equivalent to Definition 5 i.e. almost everywhere uniform stability of Λ with respect to ρ_0. Hence, we have the proof of necessity of (11). □

The next corollary shows that for systems with an almost everywhere stable attractor set Λ, the Lyapunov density can be obtained as a solution of linear partial differential equation.

Corollary 14. *Let $X \subset \mathbb{R}^n$ be compact and Γ_i defined below denote the inflow part of the boundary ∂X which is assumed to be $\mathcal{C}^2$:*

$$\Gamma_i = \{x \in \partial X\colon\ \vec{f}(x) \cdot \vec{\eta}(x) < 0\}. \tag{22}$$

Furthermore, let us assume that the attractor set Λ is almost everywhere uniformly stable with respect to the measure m with density $0 \leqslant \rho_0 \in \mathcal{D}(\mathbb{A}^1) \cap L^1(\Lambda^c)$. Then, the Lyapunov density with respect to density ρ_0 can be computed by solving the following linear partial differential equation:

$$\mathbb{A}^1 \rho(x) = -\rho_0(x), \tag{23}$$

with the following homogeneous Dirichlet boundary conditions

$$\rho|_{\Gamma_i} = 0. \tag{24}$$

Proof. The proof is the same as the proof of necessity in Theorem 13. The Dirichlet boundary condition follows by virtue of $\rho \in \mathcal{D}(\mathbb{A}^1) \cap L^1(X \setminus B_\delta)$. □

Remark 15. The solution of Eq. (23) has to be understood in the weak sense i.e., the derivatives that appear in Eq. (23) are weak derivatives. If X has no boundary or no inflow part on the boundary, then the boundary conditions (24) are not needed i.e. (23) needs to be solved without boundary conditions. We also note that if almost everywhere uniform stability of the attractor set Λ is not true, then there will be accumulation of mass on a subset $A \subset X \setminus B_\delta$ of non-zero measure which will lead to a solution $\rho \notin L^1(X \setminus B_\delta)$ i.e. a solution that is singular in $X \setminus B_\delta$ (see Fig. 4: Example 4 in Section 4).

For the case when m is the Lebesgue measure, we have $\rho_0(x) = \chi_{\Lambda^c}(x)$. We next show the relationship between almost everywhere uniform stability with respect to Lebesgue measure on Λ^c and arbitrary density $\rho_0 \in L^1(\Lambda^c)$.

Corollary 16. *The attractor set $\Lambda \subset X$ for the system of differential equation* (1) *is almost everywhere uniformly stable with respect to Lebesgue measure on Λ^c if and only if it is almost everywhere uniformly stable with respect to every finite measure m with density $0 \leqslant \rho_0 \in L^1(\Lambda^c)$.*

Proof. We have already established that almost everywhere uniform stability of Λ is equivalent to the existence of a positive solution $\rho(x) \in L^1(X \setminus B_\delta)$ to the PDE given by (23)–(24). The proof of necessity follows by choosing $\rho_0 = \chi_{\Lambda^c}$. To prove sufficiency, fix $\epsilon > 0$. We have that any function $\rho_0(x) \in L^1(\Lambda^c)$ is a strong limit (in L^1 norm) of a sequence of simple functions $\{\psi_N(x)\}_{N=0}^{\infty}$. Also, we can choose $\{\psi_N(x)\}$ to be an increasing sequence satisfying $0 \leqslant \psi_N(x) \leqslant \rho_0(x)$. We denote the sequence as follows

$$\psi_N(x) = \sum_{i=1}^{N} \lambda_i \chi_{A_i}(x),$$

where $A_i \subset \Lambda^c$. Now we have the following

$$\int_T^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1 \rho_0(x)\,dx\,dt = \int_T^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1 \psi_N(x)\,dx\,dt + \int_T^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1\big(\rho_0(x) - \psi_N(x)\big)\,dx\,dt. \tag{25}$$

First, we estimate the second term. We note that $\mathbb{P}_t^1(\rho_0(x) - \psi_N(x)) \geqslant 0$, $\forall N \in \mathbb{N}$, by property (2) applied to $\mathbb{P}_t^1$. Hence we have the following

$$\int_{X\setminus B_\delta} \mathbb{P}_t^1\big(\rho_0(x) - \psi_N(x)\big)\,dx = \big\|\mathbb{P}_t^1\big(\rho_0(x) - \psi_N(x)\big)\big\|_{L^1(X\setminus B_\delta)}.$$

Using the continuity of $\mathbb{P}_t^1$ on $L^1(X \setminus B_\delta)$ for fixed $t > 0$, there exists an $N = N_0$ large enough such that

$$\big\|\mathbb{P}_t^1\big(\rho_0(x) - \psi_{N_0}(x)\big)\big\|_{L^1(X\setminus B_\delta)} \leqslant \frac{\epsilon}{2} e^{-t}.$$

Hence we have,

$$\int_{T_0}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t^1\big(\rho_0(x)-\psi_{N_0}(x)\big)\,dx\,dt \leqslant \int_{T_0}^{\infty} \frac{\epsilon}{2}e^{-t}\,dt = \frac{\epsilon}{2}e^{-T_0} \leqslant \frac{\epsilon}{2}.$$

Hence we have,

$$\int_{T_0}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t\big(\rho_0(x)-\psi_{N_0}(x)\big)\,dx\,dt \leqslant \frac{\epsilon}{2}. \tag{26}$$

Next, we fix $N = N_0$ from the previous argument and estimate the first term:

$$\begin{aligned}\int_{T}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t^1\psi_{N_0}(x)\,dx\,dt &= \sum_{i=1}^{N_0}\lambda_i\int_{T}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{A_i}(x)\,dx\,dt \\ &= \sum_{i=1}^{N_0}\lambda_i\int_{T}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{A_i}(x)\,dx\,dt.\end{aligned}$$

Since $A_i \subset \Lambda^c$ we can apply the definition of almost everywhere stability to choose $T = T_i$, $i = 1\dots N_0$ that makes each of the N_0 terms less than or equal to $\frac{\epsilon}{2\lambda_i N_0}$. We choose $T_0 = \max_{i=1\dots N_0}(T_i)$. Hence, we have the following

$$\sum_{i=1}^{N_0}\lambda_i\int_{T_0}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{A_i}(x)\,dx\,dt \leqslant \sum_{i=1}^{N_0}\lambda_i\int_{T_0}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{\Lambda^c}(x)\,dx\,dt \leqslant \frac{\epsilon}{2}.$$

Hence we have

$$\int_{T_0}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t^1\psi_{N_0}(x)\,dx\,dt \leqslant \frac{\epsilon}{2}. \tag{27}$$

From Eqs. (26) and (27) we have

$$\int_{T_0}^{\infty}\int_{X\setminus B_\delta} \mathbb{P}_t^1\rho_0(x)\,dx\,dt \leqslant \epsilon. \tag{28}$$

This proves the corollary. □

Remark 17. We note that ρ_0 has two interpretations:

(1) We can think of ρ_0 as the Radon–Nikodym derivative of the measure with respect to which we are investigating the stability of (1).
(2) On the other hand, ρ_0 is the density for the initial mass distribution that evolves according to the non-steady state Frobenius–Perron PDE given by

$$\frac{\partial\rho}{\partial t} = \mathbb{A}^1\rho, \qquad \rho(x,0)=\rho_0(x), \qquad \rho|_{\Gamma_i}=0.$$

Since the almost everywhere stability of Λ is a property of the ODE (1), and not the initial distribution of mass, we expect that stability should be independent of the choice of $\rho_0 \in L^1(\Lambda^c)$, which is exactly what the above corollary says.

We end the theory part of the paper by illustrating two examples in one dimension which admit an explicit formula for the Lyapunov density $\rho(x)$. First, we show an example of a case which is known to be almost everywhere uniformly stable.

Example 18. We consider the following ODE:

$$\dot{x} = -x, \qquad x(0) = x_0. \tag{29}$$

It is well known that for the above system (29), the equilibrium point $x = 0$ is exponentially stable and hence also almost everywhere uniformly stable. If we choose $X = [0, 2]$, $\Lambda = \{0\}$, $B_\delta = (0, 1)$, $\rho_0(x) = \chi_{[1,2]}(x)$, then $\phi_t(x) = xe^{-t}$, $\mathbb{P}_t^1 \rho(x) = e^t$, $t \in [0, \ln(\frac{2}{x})]$ and the Lyapunov density is the solution of the following

$$\frac{d}{dx}\big(x\rho(x)\big) = -1, \qquad \rho(2) = 0, \quad x \in [1, 2]. \tag{30}$$

The Lyapunov density in this case is given by $\rho(x) = \frac{2}{x} - 1$, $x \in [1, 2]$. We note that the same solution can be obtained by explicitly calculating $\int_0^\infty \mathbb{P}_t^1 \rho_0(x)\, dt$.

Next, we show an example of a system that has an equilibrium point $x = \pi$ which is almost everywhere stable but not almost everywhere uniformly stable.

Example 19. We consider $\dot{x} = -\sin^3(x)$, with $\Lambda = \{0\}$, $X = [0, \frac{3\pi}{2}]$ and $B_\delta = (0, 1)$. It is clear that $x = 0$ is an attractor and attracts everything in X if we assume that $x = 2\pi$ and $x = 0$ are the same point on the circle. Therefore, $x = 0$ is almost everywhere stable. However, when we compute the density with $\rho_0 = \chi_{[1, \frac{3\pi}{2}]}$, we have the following

$$\frac{d}{dx}\big(\sin^3(x)\rho(x)\big) = -1, \qquad \rho\left(\frac{3\pi}{2}\right) = 0, \quad x \in \left[1, \frac{3\pi}{2}\right]$$
$$\Rightarrow \quad \rho(x) = -\frac{(x - \frac{3\pi}{2})}{\sin^3(x)}. \tag{31}$$

It is clear that $\rho(x) > 0$. However $\rho(x) \notin L^1([1, \frac{3\pi}{2}])$ due to the singularity at $x = \pi$ and hence $x = 0$ is not almost everywhere uniformly stable. The reason behind the discrepancy is because the dynamics near $x = \pi$ is extremely slow. Hence, even though $x = 0$ attracts everything, points near $x = \pi$ move very slowly and hence integrability fails.

4. Numerical simulation

This section presents some preliminary results on utilizing the theoretical developments towards formulating a computational framework that efficiently computes the Lyapunov density, ρ for a dynamical system defined by a differential equation $\dot{x} = f(x)$. Based on the proposed theory in this paper, the Lyapunov density, if it exists, is obtained as a nonnegative solution of the following partial differential equation:

$$\mathbb{A}^1 \rho(x) = -\rho_0(x) \tag{32}$$

with boundary conditions as specified in Theorem 14, where ρ_0 is the initial density with respect to which almost everywhere stability is verified. As stated earlier, this equation describes the steady state density distribution of a fluid under the action of a vector field, f, starting from an initial density distribution ρ_0. Formula (13) suggests a time-integrated approach for the computation of the Lyapunov density

$$\rho(x) = \lim_{T\to\infty} \rho(x, T) = \lim_{T\to\infty} \int_0^T \tilde{\rho}(x, t)\, dt \tag{33}$$

where $\tilde{\rho}(x, t)$ is the solution of the following linear partial differential equation:

$$\tilde{\rho}_t + \nabla \cdot (f\tilde{\rho}) = \rho_0(x), \tag{34}$$

with initial conditions $\tilde{\rho}(x, 0) = \rho_0(x)$. This equation has been particularly well studied in the fluid dynamics community since it represents problems where convection plays an important role (convection-dominated flows). Several techniques [19,20] have been developed to solve these equations in 2 and 3 dimensions. In this paper, we utilize a Finite Element Method based computation strategy to solve this first order degenerate linear partial differential equation. The Finite Element Method (FEM) is a numerical strategy for finding approximate solutions to partial differential equations [21]. The FEM method converts a partial differential equation into a set of algebraic equations by discretizing a continuous domain into a finite set of discrete sub-domains (elements). The value of the unknown variable on each sub-domain is subsequently computed. Finite element methods applied to convection-dominated equations employ stabilization to accurately capture the steep gradients and discontinuities. In this work, stabilization is established by using the popular Streamline Upwind Petrov Galerkin method (SUPG approximation) [20]. We refer the interested reader to any standard monograph discussing FEM for fluid flow for a detailed discription of stabilization techniques (e.g. [22]).

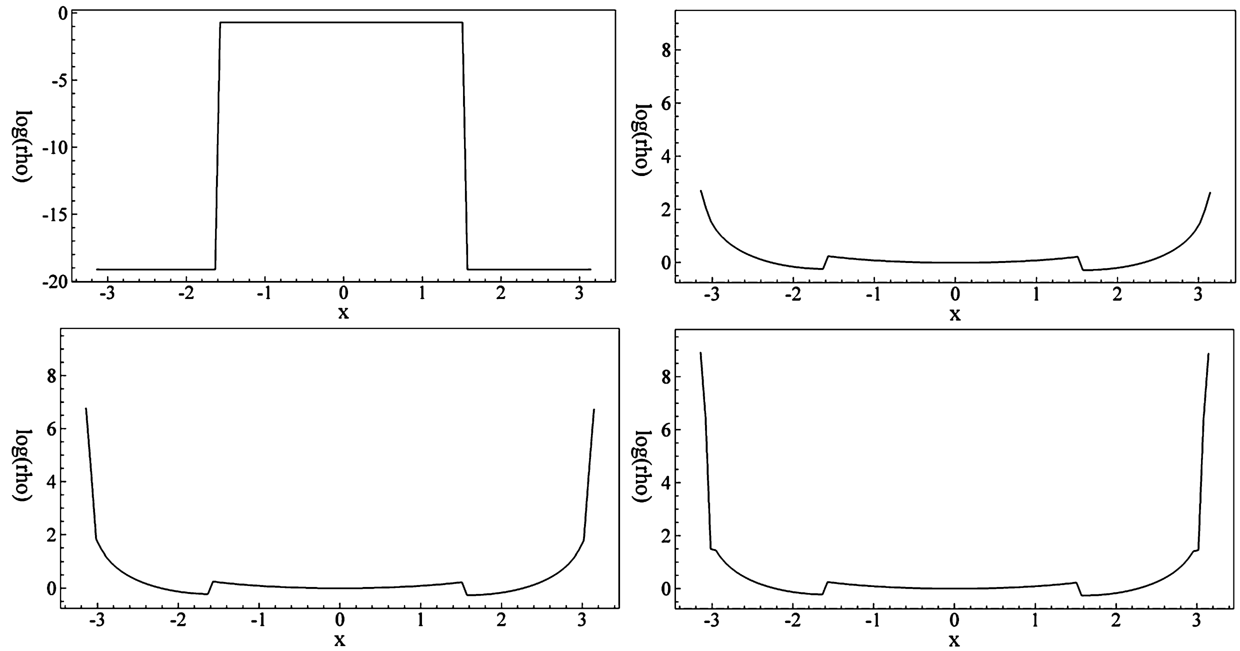

Fig. 1. The time evolution of the Lyapunov density for the $\dot{x} = \sin(x)$ system. Note the logarithmic scale used to represent ρ. The measure is concentrated towards the stable fixed points with increasing time (left to right).

4.1. Examples

We use a finite element based method to compute Lyapunov density for three different systems. Our first example is a one-dimensional system described by the following differential equation

$$\dot{x} = \sin x, \tag{35}$$

where $x \in (-\pi, \pi]$ is considered mod 2π. It is obvious that this system has two fixed points, an unstable fixed point at $x = 0$ and a stable fixed point at $x = \pi$. The stable fixed point at $x = \pi$ is almost everywhere stable. In order to verify this, we solve Eq. (32) with $\rho_0 = 1$ inside the region $[-1, 1]$ and smoothly becoming zero elsewhere. The domain $[-\pi, \pi]$ is discretized into 100 elements. Fig. 1 shows the plots of $\log(\rho(x, T))$ from Eq. (33) with increasing T. The non-negativity of the Lyapunov density verifies that almost all initial condition inside the region $[-1, 1]$ will asymptotically converge to the equilibrium point at $x = \pi$.

The second system under consideration is the Van der Pol oscillator defined as

$$\dot{\mathbf{x}} \equiv (\dot{x}, \dot{y})^T = \big(y, \big(1 - x^2\big)y - x\big)^T \equiv f(x, y). \tag{36}$$

This nonlinear oscillator is known to have a stable limit cycle, furthermore the limit cycle is almost everywhere stable. For this example we set ρ_0 to be equal to 1 in a disc of radius 4.0 and smoothly becoming zero elsewhere. The computational domain is $D = [-4,\ 4] \times [-4,\ 4]$. Zero Dirichlet conditions are applied at the boundaries of D. The domain is discretized using 100×100 isoparametric quadrilateral elements. Fig. 2 plots iso-contours of ρ. The large value of Lyapunov density in the region close to the part of the limit cycle signifies that the system trajectories spend large amount of time in this region before finally converging to the limit cycle.

The third example that we investigate is a pendulum with friction [23]. The pendulum is modeled using the following system of differential equations:

$$(\dot{x}, \dot{y}) = \big(y, -y - \sin(x)\big) \equiv f(x, y). \tag{37}$$

This system is periodic with a period of π and has a stable equilibrium point at $(0, 0)$ and unstable equilibrium point at $(\pi, 0)$. The computational domain is $[-2\pi, 2\pi] \times [-4, 4]$. We use homogeneous Dirichlet boundary conditions on the inflow portion of the boundary where the vector field f points inside. These points were identified by computing $f \cdot \eta$ and checking whether the dot product is negative. Fig. 3 shows the phase portrait and logarithm of the Lyapunov density computed using the finite element technique. It is clear from the plots that the density has a large value at points where a lot of streamlines squeeze through in the phase portrait. In [23], it has been shown that the density function, as introduced by Rantzer [2], has the property that it is zero along the stable manifold of the unstable equilibrium point at $(-\pi, 0)$ and $(\pi, 0)$. However,

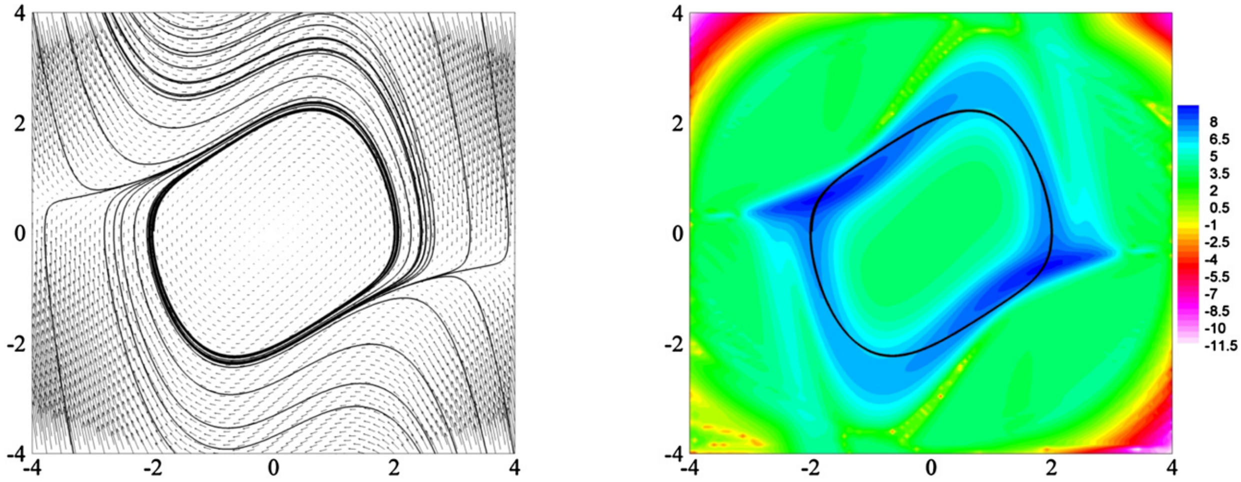

Fig. 2. The Lyapunov density for the Van der Pol oscillator computed using a stabilized finite element framework. Left: A phase portrait of the Van der Pol oscillator. Right: The corresponding Lyapunov density plot with $\log \rho$ plotted.

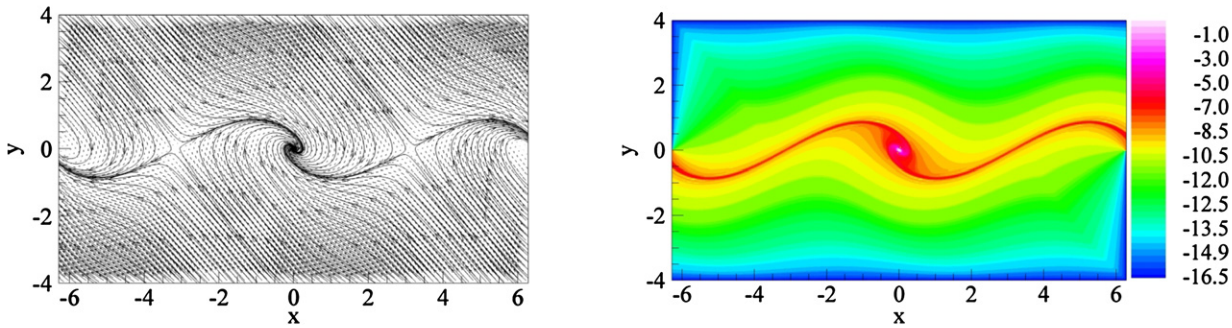

Fig. 3. Left: Phase portrait of the pendulum with friction. Right: Plot of $\log(\rho)$ for the pendulum example with friction.

from Fig. 3, we see that the Lyapunov density function as introduced in this paper is not zero along the stable manifold and furthermore is smooth in that region.

The fourth example is one that is not almost everywhere uniformly stable:

$$\dot{x} = \sin(2x), \tag{38}$$

where $x \in X = [-\pi, \pi]$. This system has two stable equilibrium points at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ and an unstable equilibrium point at $x = 0$. Neither of the stable equilibrium points is almost everywhere uniformly stable. To verify this, we again solve Eq. (32) for the system given in (38). We chose a δ neighborhood around $x = -\frac{\pi}{2}$ with $\delta = 0.1$ with a uniform initial density of $\rho_0 = 0$ inside the region $[-\frac{\pi}{2} - 0.1, -\frac{\pi}{2} + 0.1]$ and $\rho_0 = 1e-4$ everywhere else. We plot the $\rho(x) \approx \int_0^T \rho(x, T)$ for different choices of large values of T in Fig. 4. The asymmetry in the peaks is due to the imbalance in initial density since $\rho_0 = 0$, $\forall x \in [-\frac{\pi}{2} - 0.1, -\frac{\pi}{2} + 0.1]$. We see that the Lyapunov density $\rho(x)$ becomes singular near the second equilibrium point $x = \frac{\pi}{2}$ which is a part of the state space, thereby asserting the fact that $x = -\frac{\pi}{2}$ is not almost everywhere uniformly stable.

We remark that the set $\{\frac{-\pi}{2}\} \cup \{\frac{\pi}{2}\}$ however, is almost everywhere uniformly stable.

4.2. Going beyond two and three dimensions

The finite element strategy provides an efficient framework to solve for the Lyapunov measure, ρ, when the phase space is limited to two or three dimensions. In fact, any appropriate traditional strategy – spectral based, finite volume, finite difference, finite element methods – that can be utilized to solve the first order linear partial differential equation is usually limited to 2 or 3 dimensions. This is because these methods involve tessellating the phase space uniformly (or quasi-uniformly in case of adaptive variants of these methods) during the discretization. Such discretization results in the so-called 'curse-of-dimensionality': an exponential increase in the number of unknowns as the dimensionality of the phase space increases, i.e. if each dimension of the phase space is discretized into say, k, sub regions (elements, volumes, or spectral coefficients), the total number of such sub-regions would be $\mathcal{O}(k^N)$, where N is the dimensionality of the phase

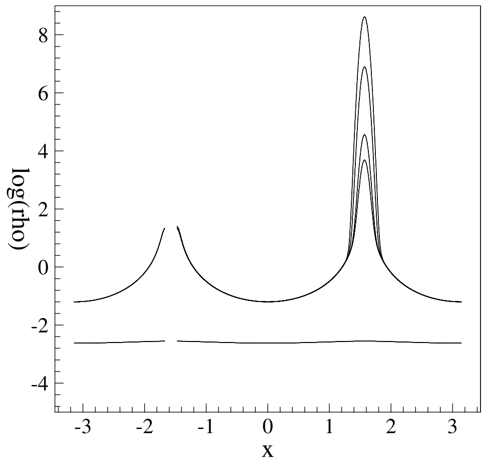

Fig. 4. The time evolution of the Lyapunov density for the $\dot{x} = \sin(2x)$ system. The height of peaks increases with increase in time.

space. This phenomena is noticable in the three examples shown above where the number of elements increased from 100 for a 1-D problem, to 10 000 for a 2D problem. Hence such methods are not computationally viable when one wants to solve problems involving high dimensional phase spaces.

One promising avenue is to look at sparse tesselations of the phase space [15]. Sparse grid collocation and interpolation strategies utilize $\mathcal{O}((\log k)^N)$ sub-regions instead of $\mathcal{O}(k^N)$, thus providing significant computational gains [16]. We are currently developing a computational framework based on this concept, which will be the focus of a forthcoming publication.

5. Conclusion

The almost everywhere uniform stability problem of nonlinear ordinary differential equation is studied using a linear partial differential equation. The notion of stability introduced in the paper is stronger than and implies the stability notion introduced in [2]. The stability certificate (Lyapunov density) belongs to a Sobolev space $\mathcal{D}(\mathbb{A}^1) \cap L^1(X \setminus B_\delta)$ which admits one order of weak differentiability. More importantly, the Lyapunov density is obtained as the solution of a linear partial differential equation which has allowed us to transfer all the intuition from linear systems theory, a mature area of research, to nonlinear systems. Establishing this connection between the stability problem of ordinary differential equation and the linear partial differential equation has provided us with a new set of computational tools for the analysis of nonlinear systems. We have used a finite element based numerical method for the computation of density in lower dimensions. Our future research efforts will focus on developing numerically efficient schemes based on sparse collocation techniques for the computation of the density in higher-dimensional systems and for making use of Lyapunov density in the design of stabilizing controller.

References

[1] H.K. Khalil, Nonlinear Systems, Prentice–Hall, New Jersey, 1996.
[2] A. Rantzer, A dual to Lyapunov's stability theorem, Systems Control Lett. 42 (2001) 161–168.
[3] S. Prajna, P.A. Parrilo, A. Rantzer, Nonlinear control synthesis by convex optimization, IEEE Trans. Automat. Control 49 (2) (2004) 1–5.
[4] R.V. Handel, Almost global stochastic stability, SIAM J. Control Optim. 45 (2006) 1297–1313.
[5] D. Angeli, An almost global notion of input to state stability, IEEE Trans. Automat. Control 49 (2004) 866–874.
[6] P. Monzón, On necessary conditions for almost everywhere stability, IEEE Trans. Automat. Control 48 (2003) 631–634.
[7] A. Rantzer, A converse theorem for density functions, in: Proceeding of IEEE Conference on Decision and Control, Las Vegas, NV, 2002, pp. 1890–1891.
[8] U. Vaidya, P.G. Mehta, Lyapunov measure for almost everywhere stability, IEEE Trans. Automat. Control 53 (2008) 307–323.
[9] U. Vaidya, P. Mehta, U. Shanbhag, Nonlinear stabilization using control Lyapunov measure, IEEE Trans. Automat. Control (2010), in press.
[10] A. Raghunathan, U. Vaidya, Optimal stabilization using Lyapunov measure, in: American Control Conference, Seattle, WA, 2008, pp. 1746–1751.
[11] U. Vaidya, R. Bhattacharya, Motion planning using navigation measure, in: Proceedings of American Control Conference, Seattle, WA, 2008, pp. 850–855.
[12] M. Dellnitz, O. Junge, Set Oriented Numerical Methods for Dynamical Systems, World Scientific, 2000, pp. 221–264.
[13] M. Dellnitz, O. Junge, On the approximation of complicated dynamical behavior, SIAM J. Numer. Anal. 36 (1999) 491–515.
[14] U. Vaidya, Converse theorem for almost everywhere stability using Lyapunov measure, in: Proceedings of American Control Conference, New York, NY, 2007, pp. 4835–4840.
[15] H.-J. Bungartz, M. Griebel, Sparse grids, Acta Numer. 14 (1982) 147–269.
[16] B. Ganapathysubramanian, N. Zabaras, Sparse grid collocation schemes for stochastic natural convection problems, J. Comput. Phys. 225 (2007) 652–685.
[17] A. Lasota, M.C. Mackey, Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics, Springer-Verlag, New York, 1994.
[18] J. Milnor, On the concept of attractor, Comm. Math. Phys. 99 (1985) 177–195.
[19] C.-W.S.B. Cockburn, Runge–Kutta discontinuous Galerkin methods for convection-dominated problems, J. Sci. Comput. 16 (3) (2001) 173–261.
[20] A.N. Brooks, T.J.R. Hughes, Streamline upwind/Petrov–Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier–Stokes equations, Comput. Methods Appl. Mech. Engrg. 32 (1982) 199–259.

[21] T.J.R. Hughes, The Finite Element Method: Linear Static and Dynamic Finite Element Analysis, Dover, New York, 2000.
[22] O.C. Zienkiewicz, R.L. Taylor, The Finite Element Method: Fluid Dynamics, Butterworth–Heinemann, Oxford, 2000.
[23] D. Angeli, Some remarks on density function dual Lyapunov methods, in: Proceedings of 42nd IEEE Conference on Decision and Control, Maui, HI, 2003, pp. 5080–5082.