A Perturbation Analysis of Parametric Resonance and Periodic Control in Spatially Distributed Systems

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***Abstract*— We consider a class of spatially distributed systems with spatially periodic coefficients that enter the system as perturbation. We then analyze the stability properties of the system for various frequencies of the perturbation. We show that, up to first order in the perturbation amplitude, parametric resonance can only occur at the isolated frequency equal to twice the frequency of the marginally-stable mode of the unperturbed system. Finally, we derive explicit expressions for the perturbation expansion of the marginally-stable mode, for all frequencies of the perturbation, and show how strict stability can be achieved for an appropriate choice of the perturbation.**

I. Introduction

It is a well-known fact in the theory of dynamical systems that the stability properties of certain ODEs can be influenced by the introduction of a parametric periodic forcing. A simple example of this is the vertical pendulum; the unstable equilibrium point of the rigid pendulum can be stabilized by the introduction of a periodic forcing at its base. This is known as *vibrational control*.

The stable equilibrium point of the non-rigid pendulum, on the other hand, can become unstable for arbitrarily small periodic perturbations of its length. These instabilities occur at particular isolated frequencies of the perturbation that are related to the parameters of the pendulum. This phenomenon is known as *parametric resonance* [1].

In the realm of partial differential equations (PDEs), little work has been done to investigate this phenomenon. Recently [2] considered the problem of parametric resonance in distributed parameter systems, where the lifting technique [3] was used to derive frequency domain representation of the periodic system, and parametric resonance found to exist in some examples.

In this paper we aim to study the occurrence of parametric resonance in spatially distributed systems. We first develop a frequency-domain representation of periodic operators using the methods of [4], with minor changes as needed for the study of spatially periodic (non-causal) operators. We then obtain, via perturbation analysis, exact conditions on the frequency of the resonating perturbation in terms of the unperturbed-system parameters. Essentially, we find those frequencies at which the unperturbed system is most "vulnerable" to a periodic perturbation, and show the asymptotic behavior of the dominant modes of the perturbed system at these and all other frequencies of the perturbation.

Our organization is as follows: We begin by briefly discussing the main results of the paper in Section II, and reviewing the mathematical preliminaries in Section III. We introduce the class of spatially distributed systems considered here in Section IV, and discuss the utilized perturbation methods in Section V. Sections V-B, V-C contain the main contributions of this work. Finally, we give some simple examples in Section VI.

Notation: We use capital letters for operators (matrices) and small letters for functions (vectors). We use the same letters for operators/functions and their Fourier transforms; the distinction should be clear from the context. $R(\lambda, T) = (\lambda - T)^{-1}$ is the resolvent of the operator T, and $\Sigma(T)$ its spectrum. $T\big|_{\mathcal{PH}}$ is the restriction of the operator T to the subspace $\mathcal{PH}$ of the space $\mathcal{H}$, where $\mathcal{P}$ is a projection operator. We use LHP to mean the left-half of the complex plane, and $k_x \in \mathbb{R}$ to denote the spatial-frequency variable. All elements of matrices that are left blank are zero.

II. Main Results

We consider systems of the form

$$\begin{aligned} \partial_t \psi(x,t) &= A_p \psi(x,t) && (1)\\ &:= \big(A \;+\; B\, z\Gamma(x)\, C\big)\psi(x,t), && x \in \mathbb{R}, \end{aligned}$$

where $\psi(x,t)$, for any given (x,t), is a vector in $\mathbb{C}^n$, A is a spatially-invariant operator, B, C are constant matrices, $\Gamma(x)$ is a spatially-periodic multiplication operator with period X and zero mean, and z is a complex number.

We also assume that the unperturbed system is marginally stable; more specifically, we assume that the Fourier symbol of A, $A(j.)$, is such that the eigenvalues of $A(jk_x)$ are in the open LHP for all $k_x \neq \pm\kappa$, and that zero is a simple eigenvalue of $A(jk_x)$ for $k_x = \pm\kappa$.

The above spatially periodic system (1) is then converted into an infinite-dimensional ODE by applying Fourier methods to the spatially periodic kernel of A_p. z is then taken to be a small parameter, and a perturbation analysis in z performed on the marginally-stable modes of the ODE. In other words, the problem is transformed

This work was supported by the Air Force Office of Scientific Research under Grant FA9550-04-1-0207 and the National Science Foundation under Grant ECS-0323814.

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into that of testing the stability of an infinite-dimensional matrix, for various values of the parameter z.

We prove that for the isolated frequency $\frac{2\pi}{X} = 2\kappa$ of the perturbation, the marginally-stable modes behave like $\pm\beta z$, whereas for all other frequencies $\frac{2\pi}{X} \neq 2\kappa$ of the perturbation, the marginally-stable modes are not affected by the perturbation up to order z^2.

An immediate implication of the above is that for the frequency $\frac{2\pi}{X} = 2\kappa$ of the perturbation, no nonzero value of z can render the system strictly stable. As a matter of fact, we show that if, for example, the B and C matrices of the system are real, then $\beta \in \mathbb{R}$, and hence all real ($z = \epsilon \in \mathbb{R}$) periodic perturbations at the resonating frequency destabilize the system, and parametric resonance occurs. On the other hand, for all other frequencies $\frac{2\pi}{X} \neq 2\kappa$ of the perturbation, we show that strict stability can always be achieved for an appropriate choice of z.

Finally, we derive explicit expressions for the perturbation expansion of the marginally-stable mode, for all possible frequencies of the perturbation. Some simple examples are also given to demonstrate the theory.

III. Preliminaries

In this section we will derive the frequency-domain representation of linear spatially periodic systems. We follow [4] with slight changes as needed for the study of spatial (non-causal) operators.

A. Frequency Response Operators

For a general linear spatial operator, we have the following integral relationship between the input $\psi(x)$ and output $\phi(x)$

$$\phi(x) = \int_{-\infty}^{\infty} G(x,\chi)\psi(\chi)d\chi, \qquad (2)$$

where $G(x,\chi)$ is called the kernel of the integral operation. In the case of a *periodic* spatial operator with period $X \in (0,\infty)$, the kernel staisfies

$$G(x+X,\chi+X) = G(x,\chi), \quad \forall\, x,\chi \in \mathbb{R}. \qquad (3)$$

One can visualize (3) as meaning that $G(x,\chi)$ is "diagonally periodic" with period X. But this can be converted into periodicity in one variable through a simple transformation.

Take $\chi = x-y$. Then by (3), $G(x+X, x+X-y) = G(x, x-y)$ for every $x, y \in \mathbb{R}$, i.e., for every given y, $G(x, x-y)$ is a periodic function of x. Hence one can write the Fourier expansion

$$G(x, x-y) = \sum_{l=-\infty}^{\infty} G_l(y) e^{j\frac{2l\pi}{X}x},$$

or, using again $\chi = x - y$,

$$G(x,\chi) = \sum_{l=-\infty}^{\infty} G_l(x-\chi) e^{j\frac{2l\pi}{X}x}.$$

Plugging this into equation (2) we get

$$\begin{aligned}\phi(x) =& \int_{-\infty}^{\infty} G(x,\chi)\psi(\chi)d\chi \\ =& \int_{-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G_l(x-\chi)e^{j\frac{2l\pi}{X}x}\psi(\chi)d\chi \\ =& \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} G_l(x-\chi)e^{j\frac{2l\pi}{X}(x-\chi)}\psi(\chi)e^{j\frac{2l\pi}{X}\chi}d\chi,\end{aligned}$$

which includes a convolution operation in x. Taking the Fourier transform of both sides, we have

$$\phi(jk_x) = \sum_{l=-\infty}^{\infty} G_l(jk_x - j\frac{2l\pi}{X})\psi(jk_x - j\frac{2l\pi}{X}).$$

Now take $k_x = \frac{2k\pi+\theta}{X}$, where $k \in \mathbb{Z}$ and $\theta \in [0,2\pi)$,

$$\phi(j\frac{2k\pi+\theta}{X}) = \sum_{l=-\infty}^{\infty} G_l(j\frac{2(k-l)\pi+\theta}{X})\psi(j\frac{2(k-l)\pi+\theta}{X})$$

which can be represented in matrix form as $\phi_\theta = \mathcal{G}_\theta\psi_\theta$ with

$$\phi_\theta := \begin{bmatrix} \vdots \\ \phi(j\frac{-2\pi+\theta}{X}) \\ \phi(j\frac{\theta}{X}) \\ \phi(j\frac{2\pi+\theta}{X}) \\ \vdots \end{bmatrix}, \quad \psi_\theta := \begin{bmatrix} \vdots \\ \psi(j\frac{-2\pi+\theta}{X}) \\ \psi(j\frac{\theta}{X}) \\ \psi(j\frac{2\pi+\theta}{X}) \\ \vdots \end{bmatrix}.$$

$$G_\theta := \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots \\ \ddots & G_0(j\frac{-2\pi+\theta}{X}) & G_{-1}(j\frac{\theta}{X}) & G_{-2}(j\frac{2\pi+\theta}{X}) & \\ \ddots & G_1(j\frac{-2\pi+\theta}{X}) & G_0(j\frac{\theta}{X}) & G_{-1}(j\frac{2\pi+\theta}{X}) & \ddots \\ & G_2(j\frac{-2\pi+\theta}{X}) & G_1(j\frac{\theta}{X}) & G_0(j\frac{2\pi+\theta}{X}) & \ddots \\ \ddots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Remark 1: Notice that ψ_θ is made up of "samples" of $\psi(jk_x)$ at the points $k_x = \frac{2k\pi+\theta}{X}$, $k \in \mathbb{Z}$. As θ runs in $[0,2\pi)$, one can interpret ψ_θ as a lifting-in-frequency of $\psi(jk_x)$, [3]. Hence one can define a unitary operator $\mathcal{M}$ such that $\psi_\theta = \mathcal{M}\psi(jk_x)$ and $\phi_\theta = \mathcal{M}\phi(jk_x)$. Then $A_\theta = \mathcal{M}A(jk_x)\mathcal{M}^*$.

Example 1: Application to spatially invariant operators. Consider the spatially invariant operator A, $\phi(x) = A\psi(x)$, with Fourier-domain representation $\phi(jk_x) = A(jk_x)u(jk_x)$, or for $k_x = \frac{2k\pi+\theta}{T}$,

$$\phi(j\frac{2k\pi+\theta}{T}) = A(j\frac{2k\pi+\theta}{T})\psi(j\frac{2k\pi+\theta}{T}).$$

This can be written in matrix form as

$$\begin{bmatrix} \vdots \\ \phi(j\frac{2k\pi+\theta}{X}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & A(j\frac{2k\pi+\theta}{X}) & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \psi(j\frac{2k\pi+\theta}{X}) \\ \vdots \end{bmatrix},$$

denoted $\phi_\theta = A_\theta \psi_\theta$. The important result here is that $A_\theta = \text{diag}\{A(j\frac{2k\pi+\theta}{T})\}$, i.e. for a spatially invariant operator A, A_θ is a block-diagonal operator.

Example 2: Application to spatially periodic multiplication operators. Consider $\phi(x) = \Gamma(x)\psi(x)$, where $\Gamma(x) = \Gamma(x + X)$ has Fourier series representation

$$\Gamma(x) = \sum_{n\in\mathbb{Z}} p_m e^{j\frac{2\pi n}{X}x}.$$

Then it can be shown that

$$\phi(j\frac{2k\pi+\theta}{X}) = \sum_{n\in\mathbb{Z}} p_{k-n}\psi(j\frac{2n\pi+\theta}{X}),$$

or in matrix form

$$\begin{bmatrix} \vdots \\ \phi(j\frac{-2\pi+\theta}{X}) \\ \phi(j\frac{\theta}{X}) \\ \phi(j\frac{2\pi+\theta}{X}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots \\ \ddots & p_0 & p_{-1} & p_{-2} & \\ \ddots & p_1 & p_0 & p_{-1} & \ddots \\ & p_2 & p_1 & p_0 & \ddots \\ \ddots & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \psi(j\frac{-2\pi+\theta}{X}) \\ \psi(j\frac{\theta}{X}) \\ \psi(j\frac{2\pi+\theta}{X}) \\ \vdots \end{bmatrix}.$$

denoted $\phi_\theta = \Gamma_\theta \psi_\theta$. Again, the important result here is that for a spatially periodic multiplication operator $\Gamma(x)$, Γ_θ is a θ-independent Toeplitz operator.

B. Spectral Analysis

The methods of the previous subsections effectively transform the question of the stability of

$$\partial_t \psi(x,t) \ = \ A_p \psi(x,t),$$

where A_p is spatially periodic operator defined on a dense domain $\mathcal{D}(A_p) \subset L^2(-\infty,\infty)$, to that of the infinite-dimensional system

$$\partial_t \psi_\theta(t) \ = \ A_{p\theta}\psi_\theta(t)$$

for all values of the variable $\theta \in [0, 2\pi)$. In such a stability analysis, we use the fact [5]

$$\Sigma(A_p) \ = \ \overline{\bigcup_{\theta\,\in\,[0,2\pi)} \Sigma(A_{p\theta})},$$

meaning, in particular, that stability properties remain preserved under such a transformation, and that a necessary condition for A_p to be stable is for $A_{p\theta}$ to be stable for all $\theta \in [0, 2\pi)$.

IV. The Perturbed System

Let us now consider a general system of the form

$$\partial_t \psi(x,t) = A_p \psi(x,t) := \big(A + B\ z\Gamma(x)\ C\big)\psi(x,t)$$

where $x \in \mathbb{R}$, and $\psi(x,t)$, for any given (x,t), is a vector in $\mathbb{C}^n$. A, B, and C are spatially-invariant operators and $\Gamma(x)$ is a spatially-periodic multiplication operator, all defined on a dense domain $\mathcal{D}(A_p) \subset L^2(-\infty,\infty)$. $\Gamma(x)$ has period X and zero mean, and z is a small complex scalar. Then, as shown in Section III-A, the representation of the system in Fourier domain would be the infinite-dimensional system

$$\partial_t \psi_\theta(t) = \big(A_\theta \ + \ zB_\theta\Gamma_\theta C_\theta\big)\psi_\theta(t).$$

In this paper, we will assume that B and C are constant operators, and that $\Gamma(x) = 2\cos(\frac{2\pi}{X}x)$. Then the operator Γ_θ will have the structure shown in Example 2, and thus

$$B_\theta\Gamma_\theta C_\theta = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & 0 & BC & & \\ & BC & 0 & \ddots & \\ & & \ddots & \ddots & \end{bmatrix}.$$

We also make the following assumptions on the symbol of A:

(a) $A(jk_x) = A(-jk_x)$,
(b) $\exists\,\kappa \neq 0$ s.t. zero is a simple eigenvalue of $A(jk_x)$ for $k_x = \pm\kappa$,
(c) $\Sigma\big(A(jk_x)\big) \subset \text{Open LHP},\ \forall\, k_x \in \mathbb{R} - \{\pm\kappa\}$.

Example 3: A scalar example of such a symbol would be $A(jk_x) = -(k_x^2 - \kappa^2)^2$.

For any given θ, define $\mathcal{A}_p(z) = \mathcal{A}_0 + z\mathcal{A}_1$ where $\mathcal{A}_0 := A_\theta$, and $\mathcal{A}_1 := B_\theta\Gamma_\theta C_\theta$ shown above. We are interested in the location of the spectrum of $\mathcal{A}_p(z)$ as the magnitude of the perturbation parameter z is increased from zero.

V. Perturbation of Spectrum

By assumption, zero is a simple eigenvalue of $A(j\kappa)$ and is not an eigenvalue of $A(jk_x)$ for $k_x \neq \kappa$. As a result, the expansion of the resolvent of $A(jk_x)$ for a given k_x around the point $\lambda = \lambda_0 = 0$ takes one of the two forms [6],[7]

$$R(\lambda, A(jk_x)) = \sum_{\sigma=0}^{\infty}(-1)^\sigma S_0(jk_x)^{\sigma+1}(\lambda-\lambda_0)^\sigma, \quad k_x \neq \kappa,$$

$$R(\lambda, A(j\kappa)) = \frac{P_0}{\lambda-\lambda_0} + \sum_{\sigma=0}^{\infty}(-1)^\sigma S_0^{\sigma+1}(\lambda-\lambda_0)^\sigma.$$

$S_0(jk_x)$ and S_0 are the reduced resolvents of $A(jk_x)$ and $A(j\kappa)$ at $\lambda = \lambda_0$, respectively. P_0 is the eigenprojection of λ_0. Notice that P_0 is a one-dimensional projection, since $\lambda_0 = 0$ is a simple eigenvalue of $A(j\kappa)$. [1]

On the other hand, since zero is a simple eigenvalue of $A(j\kappa)$, and $\mathcal{A}_0 = A_\theta = \text{diag}\{A(j\frac{2k\pi+\theta}{T})\}$, zero can only be a semi-simple eigenvalue of $\mathcal{A}_0$. Thus the

[1] One can show that for $\lambda_0 = 0$ and $k_x \neq \kappa$, $S_0(jk_x) = -A(jk_x)^{-1}$. This, together with the assumption that the real part of the eigenvalues of $A(jk_x)$, $k_x \neq \kappa$, are negative, gives that $S_0(jk_x)$ has only eigenvalues with positive real part for all $k_x \neq \kappa$.

resolvent of $\mathcal{A}_0$, around the point $\lambda = \lambda_0 = 0$, can be written as

$$R(\lambda, \mathcal{A}_0) = \frac{\mathcal{P}_0}{\lambda - \lambda_0} + \sum_{\sigma=0}^{\infty} (-1)^{\sigma} \mathcal{S}_0^{\sigma+1} (\lambda - \lambda_0)^{\sigma}. \tag{4}$$

We will discuss the structure of $\mathcal{P}_0$ and $\mathcal{S}_0$ in more detail later.

But before going any further, let us emphasize a few facts about the operator $\mathcal{A}_p(z)$. The unperturbed operator $\mathcal{A}_0 = A_\theta$ has the blocks $A(j\dfrac{2k\pi + \theta}{X})$ on its main diagonal, and zero everywhere else. Since $\mathrm{Re}\{A(jk_x)\} < 0 \quad \forall k_x \neq \kappa$, if for a given X and θ, $\kappa \neq \dfrac{2k\pi + \theta}{X}$ for any $k \in \mathbb{Z}$, then $\mathcal{A}_0$ is a strictly negative operator, and so will be $\mathcal{A}_p(z) = \mathcal{A}_0 + z\mathcal{A}_1$ for small enough z. This means that to analyze the stability properties of $\mathcal{A}_p(z)$ for a given X, one only needs to look at those values of θ for which the frequency κ is "hit" or "picked-up" by $\dfrac{2k\pi + \theta}{X}$ for some integer k_0, i.e., the singular block $A(j\kappa)$ appears on the diagonal of $\mathcal{A}_0$ at some index k_0. These are precisely the θ that are most significant in the stability of $\mathcal{A}_p$ when the "sup" is taken over all $\theta \in [0, 2\pi)$ in Section III-B, and these will be the θ we will focus on in the following perturbation analysis.

Now, notice that for any given X, there exists *at least one* $\theta \in [0, 2\pi)$ for which $\mathcal{A}_0$ has a singular block on its main diagonal. More precisely, since the frequencies $-\kappa$ and κ correspond, via $k_x = \dfrac{2k\pi + \theta}{X}$, to the two pairs (θ_-, k_-) and (θ_+, k_+), respectively,

$$-\kappa = \frac{2k_-\pi + \theta_-}{X}\ , \qquad \kappa = \frac{2k_+\pi + \theta_+}{X}\ ,$$

$\mathcal{A}_0$ will have on its diagonal, a singular block at the index k_- for $\theta = \theta_-$, or at the index k_+ for $\theta = \theta_+$.

Notice also, that for certain values of X we have $\theta_- = \theta_+$, and the singular block $A(j\kappa)$ appears *twice* on the diagonal of $\mathcal{A}_0$, namely at both indices k_- and k_+. One can show that this happens only when $\theta = \theta_- = \theta_+ = 0$, or $\theta = \theta_- = \theta_+ = \pi$. This can be used to characterize all such periods X**:** for $\theta = 0$, all periods $X = \dfrac{2\pi k}{\kappa}$, $k = 1, 2, ...$, lead to $\mathcal{A}_0$ having a pair of singular blocks at $k_- = -k$ and $k_+ = k$**;** for $\theta = \pi$, all periods $X = \dfrac{2\pi k + \pi}{\kappa}$, $k = 0, 1, 2, ...$, lead to $\mathcal{A}_0$ having a pair of singular blocks at $k_- = -k - 1$ and $k_+ = k$. For future reference, we define

$$\begin{aligned} \mathcal{X}_0 &:= \{X \mid X = \frac{2\pi k}{\kappa},\ k = 1, 2, ...\}, \\ \mathcal{X}_\pi &:= \{X \mid X = \frac{2\pi k + \pi}{\kappa},\ k = 0, 1, 2, ...\}, \\ \mathcal{X} &:= \mathcal{X}_0 \cup \mathcal{X}_\pi, \end{aligned}$$

To recap, $\mathcal{X}_0$ contains all the periods of the perturbation, at which $\mathcal{A}_0$ will have a pair of singular blocks on its diagonal for $\theta = 0$, and $\mathcal{X}_\pi$ contains all the periods at which $\mathcal{A}_0$ will have a pair of singular blocks for $\theta = \pi$.

Let us look more closely now at the structure of $\mathcal{P}_0$ and $\mathcal{S}_0$. Assume $X \notin \mathcal{X}$ and $\theta = \theta_+ \neq \theta_-$ (i.e. $\theta \neq 0$ and $\theta \neq \pi$). Then $\mathcal{A}_0$ will have only one singular block on its diagonal and

$$\mathcal{P}_0 = \begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & P_0 & \\ & & & & 0 \\ & & & & & \ddots \end{bmatrix},$$

$$\mathcal{S}_0 = \begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & S_0(j\kappa - j\frac{2\pi}{X}) & & \\ & & & S_0 & \\ & & & & S_0(j\kappa + j\frac{2\pi}{X}) \\ & & & & & \ddots \end{bmatrix}.$$

A similar structure would occur for $X \notin \mathcal{X}$ and $\theta = \theta_- \neq \theta_+$. On the other hand, for $X \in \mathcal{X}$ and $\theta = \theta_+ = \theta_-$ (i.e. $\theta = 0$ or $\theta = \pi$), $\mathcal{A}_0$ will have two singular blocks on its diagonal and

$$\mathcal{P}_0 = \begin{bmatrix} \ddots & & & & & & & \\ & 0 & & & & & & \\ & & P_0 & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & & \\ & & & & & & P_0 & \\ & & & & & & & 0 \\ & & & & & & & & \ddots \end{bmatrix},$$

$$\mathcal{S}_0 = \begin{bmatrix} \ddots & & & & & & & \\ & S_0(-j\kappa - j\frac{2\pi}{X}) & & & & & & \\ & & S_0 & & & & & \\ & & & S_0(-j\kappa + j\frac{2\pi}{X}) & & & & \\ & & & & \ddots & & & \\ & & & & & S_0(j\kappa - j\frac{2\pi}{X}) & & \\ & & & & & & S_0 & \\ & & & & & & & S_0(j\kappa + j\frac{2\pi}{X}) \\ & & & & & & & & \ddots \end{bmatrix}.$$

A. Reduction Theory

As we have seen, $\mathcal{A}_p(z)$ is an infinite-dimensional operator, and we are interested in the how the perturbation affects only the particular set $\lambda_0 = 0$ of its eigenvalues. But it is a standard result in perturbation theory that the study of a finite system of eigenvalues of an infinite-dimensional operator can be reduced to a problem in a finite-dimensional space [6].

Another obstacle in the perturbation analysis of $\mathcal{A}_p(z)$ is that $\lambda_0 = 0$ could, as we have shown, be a degenerate (repeated) eigenvalue. In this case, one can employ Reduction Theory to compute the perturbation series of $\lambda_0(z)$ until full splitting of the repeated eigenvalue occurs. Here we will only state the main result of

Reduction Theory for semi-simple eigenvalues as needed for our problem. For a general treatment see [7].

Theorem 1: The coefficients of the perturbation series $\lambda_0(z) = \lambda_0 + \lambda_0^{(1)} z + \lambda_0^{(2)} z^2 + ...$ of the eigenvalue $\lambda_0(0) = \lambda_0 = 0$ of $\mathcal{A}_p(z) = \mathcal{A}_0 + z\mathcal{A}_1$ can be found from

$$\det(\lambda_0^{(1)} - \mathcal{A}_0^{(1)}|_{\mathcal{P}_0 l^2}) = 0,$$
$$\det(\lambda_0^{(2)} - \mathcal{A}_0^{(2)}|_{\mathcal{P}_0 l^2}) = 0,$$
$$\vdots$$

where $\mathcal{A}_0^{(1)} = \mathcal{P}_0\mathcal{A}_1\mathcal{P}_0$, $\mathcal{A}_0^{(2)} = \mathcal{P}_0\mathcal{A}_1\mathcal{S}_0\mathcal{A}_1\mathcal{P}_0$, ..., and $\mathcal{P}_0$ is given by (4). The process stops once $\det(\lambda_0^{(m)} - \mathcal{A}_0^{(m)}|_{\mathcal{P}_0 l^2}) = 0$ has distinct solutions for some m, i.e., splitting of the repeated eigenvalue occurs.[2]

In the following, we have computed the first two coefficients of the perturbation series for the eigenvalue zero of $\mathcal{A}_p(z)$.

To summarize the results:

- $X \notin \mathcal{X}$
$$\lambda_0(z) = \alpha z^2 + O(z^3)$$
$$\text{simple eigenvalue}$$
- $X \in \mathcal{X}$
  - ◇ $X \in \mathcal{X}_0$
    - ⋆ $X \neq \frac{2\pi}{\kappa}$
$$\lambda_0(z) = \alpha z^2 + O(z^3)$$
$$\text{repeated eigenvalue}$$
    - ⋆ $X = \frac{2\pi}{\kappa}$
$$\lambda_0(z) = (\sigma + \delta) z^2, \quad \lambda_0(z) = (\sigma - \delta) z^2$$
$$\text{simple eigenvalues}$$
  - ◇ $X \in \mathcal{X}_\pi$
    - ⋆ $X \neq \frac{\pi}{\kappa}$
$$\lambda_0(z) = \alpha z^2 + O(z^3)$$
$$\text{repeated eigenvalue}$$
    - ⋆ $X = \frac{\pi}{\kappa}$
$$\lambda_0(z) = \beta z, \quad \lambda_0(z) = -\beta z$$
$$\text{simple eigenvalues}$$

The details of the derivations and the expressions for the scalars α, σ, δ, and β are given in the Appendix.

[2]To be precise, semi-simplicity of the eigenvalues of $\mathcal{A}_0^{(i)}$ is required at every step of the above procedure. But we will show that under the assumptions we have made in this paper, this will always be the case.

B. Parametric Resonance

A quick review of the above results reveals that perturbations with the isolated period $X_{res} := \pi/\kappa$ have a structurally different effect on the zero eigenvalue of the unperturbed system. Namely, for $X = \pi/\kappa$, $\lambda_0(z)$ is linear in z, whereas for $X \neq \pi/\kappa$, $\lambda_0(z)$ is quadratic in z.

Another important observation here is that for $X = X_{res}$ there is a 180-degree phase difference in the split eigenvalues $\lambda_0(z) = \pm\beta z$, if $\beta = P_0 BCP_0|_{P_0\mathbb{C}^n} \neq 0$. This would mean, for example, that if the B and C matrices of the system are such that $\beta \in \mathbb{R}$, then all real ($z = \epsilon \in \mathbb{R}$) periodic perturbations with period $X = X_{res}$ destabilize the perturbed system, and parametric resonance occurs.

As a final comment, note that

$$X_{res} = \frac{\pi}{\kappa} \implies \frac{2\pi}{X_{res}} = 2\kappa,$$

i.e., $\Gamma(x)$ has *twice the frequency* of the marginally-stable mode of the system, κ.

C. Stabilization

Essentially, for stabilization we try to avoid the resonating period X_{res} in the periodic gain $\Gamma(x)$, and then choose the complex gain z appropriately . For example, assume X is chosen such that $X \notin \mathcal{X}$. Then as shown above, the zero eigenvalue behaves like $\lambda_0(z) = \alpha z^2 + O(z^3)$. Clearly, one can always choose z so as to yield $\lambda_0(z)$ in the open LHP, and hence strictly stabilize the perturbed system. For instance, if $\alpha \in \mathbb{R}$, then $z = j\epsilon$, $\epsilon \in \mathbb{R}$, will stabilize the system.

VI. Examples

Example 4: Let us take the scalar system

$$A(jk_x) = -(k_x^2 - 1)^2,$$
$$B = C = 1, \quad \Gamma(x) = 2\cos(\frac{2\pi}{X} x).$$

Then $P_0 = 1$, $S_0 = 0$, and $S_0(jk_x) = \frac{1}{(k_x^2-1)^2}$. Using the expressions for α, σ, δ, and β given in the Appendix, and that $\kappa = 1$ here, we have

$$\alpha = \frac{1}{\left((1 - \frac{2\pi}{X})^2 - 1\right)^2} + \frac{1}{\left((1 + \frac{2\pi}{X})^2 - 1\right)^2},$$
$$\sigma = \frac{1}{(0-1)^2} + \frac{1}{(2^2-1)^2} = 1 + \frac{1}{9},$$
$$\delta = \frac{1}{(0-1)^2} = 1,$$
$$\beta = 1.$$

Clearly $X_{res} = \frac{\pi}{1} = \pi$.

Example 5: Consider the scalar heat equation

$$A(jk_x) = -k_x^2,$$

with B, C, and $\Gamma(x)$ the same as those in the previous example. For this system, $\kappa = 0$, and the problem is degenerate, in that $A(jk_x)$ becomes singular at only one value of the spatial frequency, $k_x = 0$. As a matter of fact, $\mathcal{A}_0$ can have at most one zero diagonal element (which occurs at index $k = 0$ for $\theta = 0$). In other words, $\lambda_0 = 0$ is a simple eigenvalue of $\mathcal{A}_0$, $\lambda_0(z)$ will never have a linear term in z, and hence parametric resonance never takes place. Clearly $P_0 = 1$, $S_0 = 0$, $S_0(jk_x) = \frac{1}{k_x^2}$, and

$$\alpha = \frac{1}{(0 - \frac{2\pi}{X})^2} + \frac{1}{(0 + \frac{2\pi}{X})^2} = 2\frac{1}{(\frac{2\pi}{X})^2},$$
$$\Longrightarrow \lambda_0(z) = 2\frac{1}{(\frac{2\pi}{X})^2} z^2 + O(z^3).$$

The above result is very interesting. It means that for every real perturbation amplitude $z \in \mathbb{R}$ the perturbed system will be unstable, where as for purely imaginary perturbations $z \in j\mathbb{R}$ the system will be stable.

VII. Conclusions and Future Work

We develop a framework suitable for the analysis of periodic spatially distributed systems, and use it to predict parametric resonance. We also show how to design a periodic perturbation to stabilize a marginally stable PDE.

Future work in this direction would include considering a broader class of marginally stable spatially distributed systems, for instance, those that have non-semisimple eigenvalues. Also, the periodic perturbation could be allowed to contain higher harmonics of the basic frequency. The effects of second order periodic perturbations could also be analyzed.

VIII. Appendix

$X \notin \mathcal{X}$

$\mathcal{A}_0^{(1)} = \mathcal{P}_0\mathcal{A}_1\mathcal{P}_0 = 0$, so [3]

$$\mathcal{A}_0^{(1)}\big|_{\mathcal{P}_0 l^2} = [0],$$

and hence $\lambda_0^{(1)} = 0$ is a simple eigenvalue. $\mathcal{A}_0^{(2)} = \mathcal{P}_0\mathcal{A}_1\mathcal{S}_0\mathcal{A}_1\mathcal{P}_0$, thus

$$\mathcal{A}_0^{(2)}\big|_{\mathcal{P}_0 l^2} = [\alpha].$$

$X \in \mathcal{X}_0$

$\mathcal{A}_0^{(1)} = \mathcal{P}_0\mathcal{A}_1\mathcal{P}_0 = 0$, so

$$\mathcal{A}_0^{(1)}\big|_{\mathcal{P}_0 l^2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and hence $\lambda_0^{(1)} = 0$ is a repeated semi-simple eigenvalue. $\mathcal{A}_0^{(2)} = \mathcal{P}_0\mathcal{A}_1\mathcal{S}_0\mathcal{A}_1\mathcal{P}_0$, and we have two cases here: for $X \neq \frac{2\pi}{\kappa}$

$$\mathcal{A}_0^{(2)}\big|_{\mathcal{P}_0 l^2} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}.$$

and for $X = \frac{2\pi}{\kappa}$

$$\mathcal{A}_0^{(2)}\big|_{\mathcal{P}_0 l^2} = \begin{bmatrix} \sigma & \delta \\ \delta & \sigma \end{bmatrix}.$$

$X \in \mathcal{X}_\pi$

For $X \neq \frac{\pi}{\kappa}$, $\mathcal{A}_0^{(1)} = \mathcal{P}_0\mathcal{A}_1\mathcal{P}_0 = 0$, which means,

$$\mathcal{A}_0^{(1)}\big|_{\mathcal{P}_0 l^2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and hence $\lambda_0^{(1)} = 0$ is a repeated semi-simple eigenvalue. $\mathcal{A}_0^{(2)} = \mathcal{P}_0\mathcal{A}_1\mathcal{S}_0\mathcal{A}_1\mathcal{P}_0$ and we have

$$\mathcal{A}_0^{(2)}\big|_{\mathcal{P}_0 l^2} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}.$$

For $X = \frac{\pi}{\kappa}$

$$\mathcal{A}_0^{(1)}\big|_{\mathcal{P}_0 l^2} = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}.$$

where

$$\alpha = P_0 BC\big(S_0(j\kappa - j\frac{2\pi}{X}) + S_0(j\kappa + j\frac{2\pi}{X})\big)BCP_0\big|_{P_0\mathbb{C}^n}$$
$$\sigma = P_0 BC\big(S_0(j0) + S_0(j2\kappa)\big)BCP_0\big|_{P_0\mathbb{C}^n}$$
$$\delta = P_0 BCS_0(j0)BCP_0\big|_{P_0\mathbb{C}^n}$$
$$\beta = P_0 BCP_0\big|_{P_0\mathbb{C}^n}$$

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[3] Wherever there is a possibility of ambiguity, we use [0] to mean the scalar zero, as opposed to a zero matrix.