Identification of sparse communication graphs in consensus networks

Fu Lin, Makan Fardad, and Mihailo R. Jovanović

Abstract— **We consider the design of distributed controller architectures for undirected networks of single-integrators. In the presence of stochastic disturbances, we identify communication topologies that balance the variance amplification of the network with the number of communication links. This is achieved by solving a parameterized family of sparsity-promoting optimal control problems whose solution traces the optimal tradeoff curve that starts at the centralized controller and ends at the controller with sparse communication links. We show that the optimal control problem can be formulated as a semidefinite program whose global solution can be computed efficiently. An example is provided to illustrate the utility of the developed approach.**

Index Terms— **Communication graphs, consensus, controller architectures, convex optimization, ℓ_1 minimization, network design, semidefinite program.**

I. Introduction

An important question in the design of networks of dynamical systems is the selection of distributed controller architectures. It is of interest to identify controller architectures that strike a balance between the performance of the interconnected system and the number of communication links. The interconnection patterns in the corresponding optimal control problem are typically described by a graph, with applications ranging from coordinated control of multi-agent systems, to average consensus in sensor networks [1]–[10]. For *fixed* topologies, several research efforts have focused on characterizing performance limitations in large networks [2], [3], [5]–[8], [11], [12]. Recently, a number of authors have considered the problem of network topology modification to improve performance of networked systems [13]–[18].

In this paper, we consider the design of undirected consensus networks in the presence of stochastic disturbances. We identify sparse communication graphs that strike a balance between the variance amplification of the network and the number of communication links. We solve a parameterized family of sparsity-promoting optimal control problems whose solution provides the optimal tradeoff curve that starts at the centralized controller and ends at the controller with a sparse communication graph. We show that the optimal control problem can be formulated as a semidefinite program (SDP) and thus, can be solved efficiently for small and medium problems.

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The paper is organized as follows. We formulate the sparsity-promoting optimal control problem for the design of single-integrator consensus networks in Section II. We obtain an SDP formulations for the optimal control problem and provide an illustrative example in Sections III and IV, respectively. We conclude the paper with a summary of our contributions in Section V.

II. Design of undirected networks of single-integrators

We consider a network with N single-integrators,

$$\dot{x}_i = u_i + d_i, \qquad i = 1, \ldots, N$$

where u_i is the control input acting on node i, and d_i is the white stochastic disturbance with zero-mean and unit variance. Each node forms its control action using a weighted sum of the differences between itself and other nodes,

$$u_i = -\sum_{j \neq i} F_{ij}\,(x_i - x_j). \tag{1}$$

We focus on connected undirected networks; thus,

$$F_{ij} = F_{ji}, \quad i \neq j.$$

A nonzero element F_{ij} corresponds to an edge between node i and node j. Thus, the communication architecture of the network is determined by the sparsity pattern of the matrix F; in particular, the number of communication links is determined by the number of nonzero elements of F.

We are interested in identifying sparsity patterns of F that strike a balance between the number of communication links and a performance measure that quantifies the consensus of the stochastically forced network. As shown in [19], this optimal control problem can be formulated as

$$\underset{F}{\text{minimize}} \quad J(F) + \gamma\,\mathbf{card}\,(F)$$

where J is the objective function defined in Section II-A and $\mathbf{card}\,(\cdot)$ is the cardinality function that counts the *number of nonzero elements* of a matrix. Larger values of the nonnegative scalar γ encourage sparser feedback gains.

A. Performance measure J

We next define the performance measure J for the consensus of the network in the presence of stochastic disturbances. Putting states of nodes, control inputs, and disturbances into vectors, e.g., $u = [\,u_1 \cdots u_N\,]^T \in \mathbb{R}^N$, yields

$$\dot{x} = -Fx + d. \tag{2}$$

For undirected networks, F is a symmetric matrix, $F = F^T$. Thus, the stability of the closed-loop system (2) amounts to the positive definiteness of the feedback gain matrix, $F \succ 0$. However, using only relative information exchange (1), the control input u does not stabilize the average mode [4]

$$\bar{x}(t) := \frac{1}{N} \sum_{i=1}^{N} x_i(t) = \frac{1}{N} \mathbb{1}^T x(t).$$

This can be verified by noting that the matrix F has a zero eigenvalue associated with the vector of all ones $\mathbb{1}$

$$F \mathbb{1} = 0.$$

For connected networks, all other eigenvalues of F are positive; hence

$$F + \mathbb{1}\mathbb{1}^T/N \succ 0.$$

In the absence of stochastic disturbances, the states of all nodes in a connected network converge to the average of the initial condition [1]

$$\lim_{t \to \infty} x(t) = \mathbb{1}\, \bar{x}(0) = \frac{1}{N} \mathbb{1}\mathbb{1}^T x(0).$$

In the presence of disturbances, however, the average mode $\bar{x}$ undergoes a random walk and its variance becomes unbounded asymptotically [2].

Several performance outputs that render the average mode $\bar{x}$ unobservable can be employed to quantify the performance of consensus [6]. Let such a performance output be given by

$$y = Qx$$

where the positive semidefinite matrix $Q = Q^T \succeq 0$ satisfies

$$Q \mathbb{1} = 0$$

(i.e., it has a zero eigenvalue associated with $\mathbb{1}$) and all other eigenvalues of Q are positive. Then the average mode associated with the eigenvector $\mathbb{1}$ is unobservable from y.

The control objective is to keep the performance output

$$z = \begin{bmatrix} z_1 \\ u \end{bmatrix} = \begin{bmatrix} Q^{1/2} \\ -F \end{bmatrix} x$$

small in the presence of stochastic disturbances d. We consider the $\mathcal{H}_2$ norm of the transfer function from d to z. Thus, the objective function J is determined by

$$J(F) = \operatorname{trace}(P) \tag{3}$$

where the observability Gramian P is the solution to the Lyapunov equation

$$(-F)P + P(-F) = -(Q + FF).$$

B. Convex relaxation of the cardinality function

Since the cardinality function is a nonconvex function of its argument, we employ the weighted ℓ_1 norm proposed in [20] as a convex relaxation

$$g(F) = \sum_{i,j} W_{ij} \, |F_{ij}|.$$

Thus, the sparsity-promoting optimal control problem for the undirected network of single-integrators is given by

$$\underset{F}{\text{minimize}} \quad J(F) + \gamma\, g(F). \tag{4}$$

We follow [20] and set the weights W_{ij} to be inversely proportional to the magnitude of the solution $F^\star$ of (4) at the previous value of γ[1],

$$W_{ij} = \frac{1}{|F_{ij}^\star| + \varepsilon}.$$

This scheme places larger weights on smaller feedback gains and consequently, these feedback gains are more likely to be dropped in the next round of iterations that are used to solve the sparsity-promoting problem (4). Here, $\varepsilon = 10^{-3}$ is introduced to have well-defined weights when $F_{ij}^\star = 0$.

III. Semidefinite programming formulation

In this section, we show that the sparsity-promoting optimal control problem (4) for the network of single-integrators can be formulated as an SDP. To this end, note that both Q and F are positive semidefinite matrices with $Q\mathbb{1} = 0$ and $F\mathbb{1} = 0$. Furthermore, for connected networks, F is a positive definite matrix when restricted to the subspace $\mathbb{1}^\perp$ (i.e., the subspace orthogonal to $\mathbb{1}$). Thus, using [10, Lemma 1], the $\mathcal{H}_2$ norm J in (3) can be written as

$$J(F) = \frac{1}{2} \operatorname{trace}\left(F^\dagger (Q + FF)\right)$$

where $F^\dagger$ denotes the Moore-Penrose pseudoinverse of F. Using the identity $FF^\dagger F = F$ and the identity [10, Lemma 1]

$$\operatorname{trace}\left(F^\dagger Q\right) = \operatorname{trace}\left((F + \mathbb{1}\mathbb{1}^T/N)^{-1} Q\right)$$

we obtain

$$J(F) = \frac{1}{2} \operatorname{trace}\left((F + \mathbb{1}\mathbb{1}^T/N)^{-1} Q + F\right). \tag{5}$$

Proposition 1: Suppose that $Q\mathbb{1} = 0$ and $Q + \mathbb{1}\mathbb{1}^T/N \succ 0$. Then the optimization problem

$$\begin{array}{ll} \underset{F}{\text{minimize}} & \frac{1}{2} \operatorname{trace}\left((F + \mathbb{1}\mathbb{1}^T/N)^{-1} Q + F\right) \\ \text{subject to} & F\mathbb{1} = 0, \quad F + \mathbb{1}\mathbb{1}^T/N \succ 0 \end{array} \tag{6}$$

can be formulated as an SDP

$$\begin{array}{ll} \underset{X,\,F}{\text{minimize}} & \frac{1}{2} \operatorname{trace}(X + F) \\ \text{subject to} & \begin{bmatrix} X & Q^{1/2} \\ Q^{1/2} & F + \mathbb{1}\mathbb{1}^T/N \end{bmatrix} \succeq 0 \\ & F\mathbb{1} = 0. \end{array} \tag{7}$$

Proof: See Appendix A. ■

We now state the main result of this section.

Proposition 2: For the objective function J in (5), the sparsity-promoting optimal control problem

$$\underset{F}{\text{minimize}} \;\; J(F) + \gamma \sum_{i,j} W_{ij} \, |F_{ij}| \tag{8}$$

[1]For $\gamma = 0$, the solution to (4) does not depend on the weighted ℓ_1 norm.

can be formulated as an SDP

$$\begin{array}{ll}\underset{X,\,Y,\,F}{\text{minimize}} & \frac{1}{2}\,\text{trace}\,(X\,+\,F)\,+\,\gamma\,\mathbb{1}^TY\mathbb{1} \\ \text{subject to} & \left[\begin{array}{cc} X & Q^{1/2} \\ Q^{1/2} & F+\mathbb{1}\mathbb{1}^T/N \end{array}\right] \succeq 0 \\ & F\,\mathbb{1}\,=\,0 \\ & -Y\,\leq\,W\circ F\,\leq\,Y. \end{array} \tag{9}$$

Proof: Transforming the weighted ℓ_1 norm in (8) to a linear function with linear inequality constraints yields

$$\begin{array}{ll}\underset{Y,\,F}{\text{minimize}} & J(F)\,+\,\gamma\,\mathbb{1}^TY\mathbb{1} \\ \text{subject to} & -Y\leq W\circ F\leq Y\end{array}$$

where Y is a matrix with nonnegative elements, $Y \geq 0$, and $\circ$ is the elementwise multiplication of matrices. The result then follows from Proposition 1. ∎

A. Solving the structured $\mathcal{H}_2$ problem: Polishing step

After identifying the sparsity pattern $\mathcal{S}$ from the solution to (9), we next turn to the $\mathcal{H}_2$ problem subject to structural constraints on the feedback matrix,

$$\begin{array}{ll}\underset{F}{\text{minimize}} & J(F) \\ \text{subject to} & F\,\mathbb{1}\,=\,0,\quad F+\mathbb{1}\mathbb{1}^T/N\,\succ\,0,\quad F\in\mathcal{S}.\end{array} \tag{10}$$

Here, we fix the sparsity pattern $F \in \mathcal{S}$ and then solve (10) to obtain the optimal feedback gain that belongs to $\mathcal{S}$. This *polishing* step can improve the performance of sparse feedback gains resulting from the SDP (9).

Let $I_{\mathcal{S}}$ be the structural identity of $\mathcal{S}$ with its ijth entry defined as

$$[I_{\mathcal{S}}]_{ij}\;=\;\begin{cases}1, & \text{if } F_{ij} \text{ is a free variable} \\ 0, & \text{if } F_{ij}=0 \text{ is required.}\end{cases}$$

Then problem (10) can be formulated as the following SDP

$$\begin{array}{ll}\underset{X,\,F}{\text{minimize}} & \frac{1}{2}\,\text{trace}\,(X\,+\,F) \\ \text{subject to} & \left[\begin{array}{cc} X & Q^{1/2} \\ Q^{1/2} & F+\mathbb{1}\mathbb{1}^T/N \end{array}\right] \succeq 0 \\ & F\,\mathbb{1}\,=\,0 \\ & F\circ I_{\mathcal{S}}\,=\,F.\end{array}$$

IV. An example

For $N = 30$ randomly distributed nodes in a region of 10×10 units, let two nodes be neighbors if their Euclidean distance is not greater than 2.5 units; see Fig. 1a. We consider the following global and local performance errors [6]:

- The global error quantifies the deviation of each node's state from the average mode

$$(z_g)_i\,=\,x_i\,-\,\bar{x},\quad i\,=\,1,\ldots,N.$$

- The local error quantifies the difference between the states of neighboring nodes

$$(z_l)_{ij}\,=\,x_i\,-\,x_j\;\text{ for }\;(i,j)\,\in\,\mathcal{E}$$

  where $\mathcal{E}$ is the edge set with (i, j) denoting an edge between two neighboring nodes.

Thus, the performance output is given by

$$z\;=\;\left[\begin{array}{c} z_g \\ z_l \\ u\end{array}\right]\;=\;\left[\begin{array}{c}(I\,-\,\mathbb{1}\mathbb{1}^T/N)\,x \\ E^Tx \\ -\,Fx\end{array}\right].$$

where E denotes the incidence matrix of the edge set $\mathcal{E}$. Each column of E is a vector of N elements representing an edge in $\mathcal{E}$; for an edge (i, j), the corresponding column of E has 1 and -1 at the ith and jth elements, and zero everywhere else. With the above choice of z, the matrix Q in (5) is determined by

$$Q\,=\,EE^T\,+\,I\,-\,\mathbb{1}\mathbb{1}^T/N.$$

We solve the sparsity-promoting optimal control problem (4), followed by the polishing step described in Section III-A, with 100 logarithmically-spaced points for $\gamma \in [10^{-3}, 1]$. As shown in Fig. 2, the number of nonzero elements of F decreases and the $\mathcal{H}_2$ norm J increases with γ. For $\gamma = 1$, the identified communication graph establishes long-range links between selected pairs of remote nodes, in addition to the interactions between neighbors; see Fig. 1b. Relative to the centralized gain F_c, the identified sparse gain F uses 12.4% nonzero elements, i.e.,

$$\textbf{card}\,(F)/\textbf{card}\,(F_c)\,=\,12.4\%$$

and achieves a performance loss of only 13.1%, i.e.,

$$(J\,-\,J_c)/J_c\,=\,13.1\%.$$

Here, F_c is the solution to (9) with $\gamma = 0$ and F is the solution to (9) with $\gamma = 1$, followed by the polishing step in Section III-A.

V. Concluding Remarks

In this paper, we consider the design of undirected consensus networks of single-integrators. We show that the sparsity-promoting optimal control problem can be formulated as an SDP. By solving a parameterized family of SDPs, we obtain a tradeoff curve between the variance amplification of the consensus network and the number of communication links in the distributed controllers. An illustrative example is provided to demonstrate the utility of the developed approach.

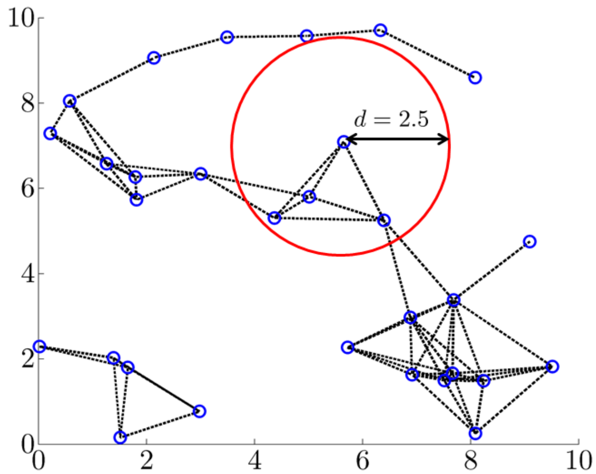
Appendix

A. Proof of Proposition 1

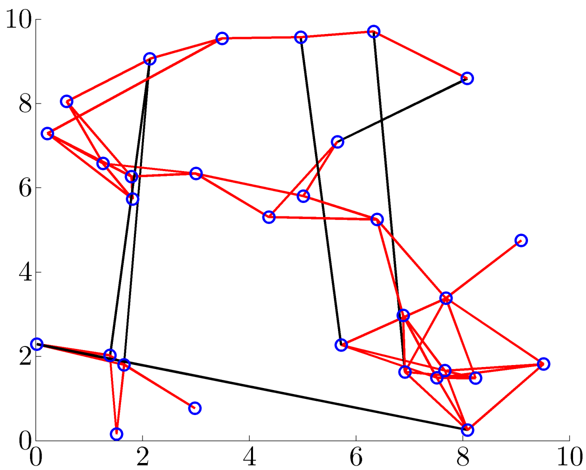
From the LMI in (7), it follows that $\bar{F} := F+\mathbb{1}\mathbb{1}^T/N \succeq 0$. We next show that $\bar{F}$ is positive definite, $\bar{F} \succ 0$. Using the generalized Schur complement [21, Appendix A.5.5], we have

$$(I-\bar{F}\bar{F}^{\dagger})Q^{1/2}\,=\,0 \tag{11}$$

where $\bar{F}^{\dagger}$ is the Moore-Penrose pseudoinverse of $\bar{F}$. Consider the spectral decomposition $Q = U\Lambda U^T$ where $U = [\,\frac{1}{\sqrt{N}}\mathbb{1}\quad V\,]$ is the orthonormal matrix and $\Lambda = \text{diag}\,(\lambda)$ with $\lambda = [\,0\ \lambda_2\ \cdots\ \lambda_N\,]$ and $\lambda_i > 0$ for $i = 2,\ldots,N$. Then multiplying U^T from the left and U from the right to (11)

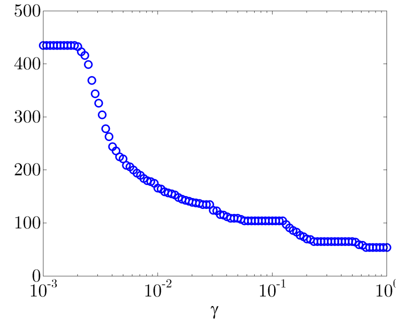


(a)

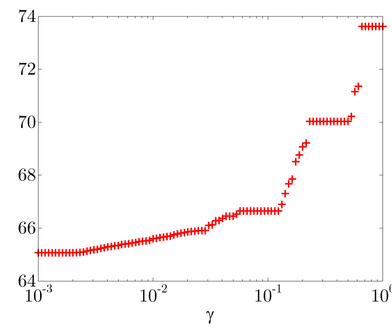


(b) $\gamma = 1$

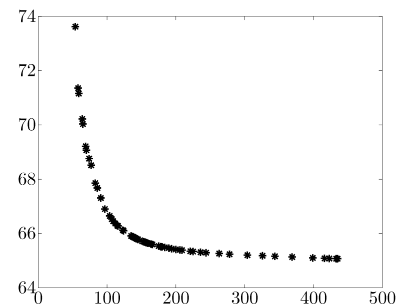
Fig. 1: (a) Local performance graph where edges connect every pair of nodes with a distance not greater than 2.5 units. (b) Identified communication graph for $\gamma = 1$, where the long-range communication links are highlighted in black color.



(a) The cardinality of F



(b) The $\mathcal{H}_2$ norm J



(c) $\textbf{card}\,(F)$ vs. J

Fig. 2: The solution to (4) as a function of γ, followed by the polishing step in Section III-A, for the network shown in Fig. 1a.

yields

$$U^T(I - \bar{F}\bar{F}^\dagger)U\Lambda^{1/2} \;=\; 0.$$

It follows that the symmetric matrix $U^T(I - \bar{F}\bar{F}^\dagger)U$ is a diagonal matrix with its diagonal equal to 0 from the 2nd to the Nth entry,

$$U^T(I - \bar{F}\bar{F}^\dagger)U \;=\; \text{diag}\,([\,a\;0\;\cdots\;0\,])$$

and thus,

$$\bar{F}\bar{F}^\dagger \;=\; I - (a/N)\mathbb{1}\mathbb{1}^T$$

where the scalar a is to be determined. We note that $a \neq 1$, because otherwise $\bar{F}\bar{F}^\dagger = I - \mathbb{1}\mathbb{1}^T/N$ implies that the range space of $\bar{F}$ is orthogonal to $\mathbb{1}$ (i.e., $\bar{F}\mathbb{1} = 0$), which leads to the contradiction

$$0 \;=\; \bar{F}\mathbb{1} \;=\; (F + \mathbb{1}\mathbb{1}^T/N)\mathbb{1} \;=\; \mathbb{1}.$$

Since $I - (a/N)\mathbb{1}\mathbb{1}^T$ is not invertible for any $a \neq 1$, we conclude that $\bar{F}$ is of full rank. Therefore, $\bar{F} \succ 0$ and $\bar{F}\bar{F}^\dagger = I$; thus, $a = 0$. Then the equivalence between (6) and (7) can be established by noting that

$$\begin{bmatrix} X & Q^{1/2} \\ Q^{1/2} & \bar{F} \end{bmatrix} \succeq 0 \iff X \succeq Q^{1/2}\bar{F}^{-1}Q^{1/2}$$

whenever $\bar{F} \succ 0$. To minimize the objective function J in (5) for $\bar{F} \succ 0$, we simply take $X = Q^{1/2}\bar{F}^{-1}Q^{1/2}$, which yields the objective function in (7).

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