On the optimal localized feedback design for multi-vehicle systems

Fu Lin, Makan Fardad, and Mihailo R. Jovanović

***Abstract*— We consider the design of localized feedback gains using relative information exchange between vehicles. The optimal controller is obtained by minimizing the global performance measure that quantifies the coherence of the large-scale network. For undirected connected graphs we show convexity of this optimal control problem, implying that its global solution can be computed efficiently. Moreover, we determine analytically the optimal localized gains for several graphs. This allows us to quantify scaling of the performance measure with the network size and to identify graphs that are favorable for maintaining coherence of the network. Another contribution of the paper lies in the characterization of all stabilizing localized feedback gains. This characterization can be utilized to examine the interplay between the underlying communication topology and the dynamics of the closed-loop system.**

***Index Terms*— Convex optimization, large-scale networks, local feedback design, multi-vehicle systems, undirected graphs.**

I. Introduction

There is a broad interest in coordination of multi-vehicle systems. Applications of such systems include formations of unmanned aerial vehicles, automated highway systems, and deployment of mobile robotic agents. Although each of these problems has its own specific challenges, several common features can be observed. In most cases, dynamically decoupled vehicles become coupled through the joint objective they are trying to achieve. The control decision for each vehicle must be made using only limited information from a subset of vehicles. This constraint on flow of information imposes fundamental limitations in the control of these systems. Recent research effort has focused on understanding the interplay between the underlying network topology and key system-theoretic properties that these networks exhibit [1]–[4].

In many applications, it is desired to have control strategies that rely only on *local* information exchange. For example, in automated highways, all vehicles can be equipped with ranging devices, allowing them to measure relative distances with respect to their immediate neighbors. It is this type of information exchange that we consider in this paper. The notion of coherence of large-scale networks with relative information exchange was introduced in [3]. For vehicular formations, this notion quantifies closeness of formation to a rigid frame by examining how deviation from average scales with the number of vehicles. It was shown in [3] that local feedback cannot maintain large-scale coherent formations in one and two spatial dimensions.

While the authors of [3] focused on regular lattices and tori, by making use of graph-theoretic tools [4], [5], we study coherence for undirected connected graphs. In particular, we formulate the problem of designing localized feedback gains as an optimal control problem with both microscopic and macroscopic performance measures as the objective function [3]. We demonstrate convexity of this problem, implying that the global optimizer can be computed efficiently [6]. Moreover, for path, star, circle, and complete graphs, we determine analytically the optimal feedback gains. These are used to obtain the asymptotic scaling of the performance measures with the network size and to identify graphs that are favorable for maintaining coherent networks. Furthermore, we characterize all stabilizing feedback gains for undirected graphs with relative information exchange. This characterization can enhance the understanding of the interplay between the underlying communication topology and the dynamics of the closed-loop system.

Our presentation is organized as follows. We borrow tools from graph theory and formulate the optimal control problem in Section II. The main results on the stability of the closed-loop system and the convexity of the design problem are derived in Section III. We then determine optimal feedback gains analytically for several aforementioned graphs in Section IV. Finally, we summarize our results and offer outlook for future research directions in Section V.

II. Problem formulation

Let $\mathcal{G}$ be an undirected connected graph with the vertex set $\mathcal{V} = \{1, \ldots, N\}$ and the edge set $\mathcal{E}$, where each edge $(i,j) \in \mathcal{E}$ is an unordered pair of distinct vertices. Each node is modeled by the single-integrator dynamics

$$\dot{x}_i \,=\, d_i \,+\, u_i, \quad i \in \mathcal{V}$$

where d_i is a zero-mean, unit-variance white stochastic disturbance and u_i is the control input. Each node has access to the *relative* information between itself and its neighbors

$$y_{ij} \,=\, x_i - x_j, \quad j \in \mathcal{N}_i$$

with $\mathcal{N}_i = \{j | (i,j) \in \mathcal{E}\}$. The control input at the ith node is then given by

$$u_i \,=\, -\sum_{j \,\in\, \mathcal{N}_i} k_{ij}(x_i \,-\, x_j),$$

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where the local feedback gains $\{k_{ij}\}$ are the design parameters. For undirected graphs, we have

$$k_{ij} \;=\; k_{ji}, \quad \text{for all} \quad i \in \mathcal{V}, \quad j \in \mathcal{N}_i.$$

The control input in vector form is $u = -Lx$, where the *weighted graph Laplacian* L is a symmetric matrix of a particular structure determined by the graph.

The weighted Laplacian can be represented using the incidence matrix of the graph [7], [8]. We associate with edge (i,j) a vector $e_{ij} \in \mathbb{R}^N$ that has 1 and -1 as the ith and jth entries, respectively, and 0 otherwise. Enumerating the edges $(i,j) \in \mathcal{E}$ by $n \in \{1,\dots,|\mathcal{E}|\}$, where $|\mathcal{E}|$ is the number of edges, we have [7], [8]

$$L \;=\; \sum_{n=1}^{|\mathcal{E}|} k_n e_n e_n^T \;=\; EKE^T$$

where

$$E \;=\; \left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_{|\mathcal{E}|} \end{array}\right] \in \mathbb{R}^{N \times |\mathcal{E}|}$$

is the *incidence matrix* of the graph, and $K \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{E}|}$ is a diagonal matrix with its main diagonal determined by $\{k_n\}_{n=1}^{|\mathcal{E}|}$. The closed-loop system is thus given by

$$\begin{array}{rcl} \dot{x} &=& -EKE^T x \,+\, d \\ z &=& H_s\, x, \end{array} \tag{1}$$

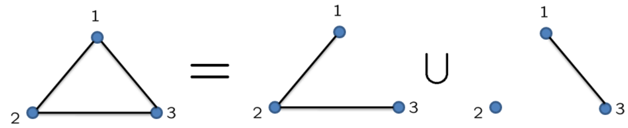
where z is the performance output penalizing the microscopic (local) and macroscopic (global) errors [3]:

- **Local error:** $z_{ij} = x_i - x_j$ for all $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$. This is a measure of the difference between neighboring nodes, with z in (1) given by
$$z \;=\; H_l\, x \;=\; E^T x.$$
- **Global error (deviation from average):** $z_i = x_i - \bar{x}$, where
$$\bar{x} \;:=\; \frac{1}{N}\sum_{i=1}^{N} x_i$$
is the *average-mode* [3]. In matrix form,
$$z \;=\; H_g\, x \;=\; \left(I \,-\, (1/N)\mathbf{1}\mathbf{1}^T\right) x,$$
where $\mathbf{1}$ is the vector with all entries 1.

It is well-known [3], [4], [7] that the average-mode $\bar{x}$ of the closed-loop system (1) is not asymptotically stable. In particular, since $E^T\mathbf{1} = 0$, the weighted Laplacian L has a zero eigenvalue associated with the eigenvector $\mathbf{1}$. However, $\mathbf{1}$ is also in the null-space of the performance output matrix H_s (with $s = l$ or $s = g$), implying that the average-mode is *unobservable* from z. We consider the design of the diagonal feedback gain K that minimizes the H_2 norm of the transfer function from disturbance d to

$$\eta \;=\; \left[\begin{array}{cc} q^{1/2}\, z^T & r^{1/2}\, u^T \end{array}\right]^T.$$

This generalized performance output introduces penalty on both states and control as is commonly done in quadratic optimal control problems, where q and r are positive scalars.



$$\left[\begin{array}{rrr} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right] \qquad \left[\begin{array}{rr} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{array}\right] \qquad \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]$$

Fig. 1: Decomposition into the tree subgraph and remaining edge, with the corresponding partition of the incidence matrix.

A. Similarity transformation

Following [4], we next introduce a similarity transformation to separate the unobservable average-mode. We note that the incidence matrix E can be represented as

$$E \;=\; \left[\begin{array}{cc} E_t & E_c \end{array}\right],$$

where E_t and E_c are the incidence matrices of a tree subgraph $\mathcal{T}$ and the remaining edges $\mathcal{C}$ (if any); see Fig. 1 for an illustration. Adding any edge in $\mathcal{C}$ to $\mathcal{T}$ forms a cycle. As a result, each column of E_c is a linear combination of the columns of E_t [4]. Therefore, E_c is in the range of the projection matrix

$$\Pi \;=\; E_t(E_t^T E_t)^{-1} E_t^T,$$

that is, $E_c = \Pi E_c$. Furthermore, we can write $E_c = E_t \Gamma$ where

$$\Gamma \;=\; (E_t^T E_t)^{-1} E_t^T E_c.$$

Thus, $E = E_t R$ with $R := \left[\begin{array}{cc} I & \Gamma \end{array}\right]$.

Following [4], we consider the similarity transformation $x = S\,\phi$, with $S = \left[\begin{array}{cc} E_t\,(E_t^T E_t)^{-1} & \mathbf{1} \end{array}\right]$ and

$$S^{-1} \;=\; \left[\begin{array}{c} E_t^T \\ (1/N)\mathbf{1}^T \end{array}\right], \quad \phi \;=\; \left[\begin{array}{c} \psi \\ \bar{x} \end{array}\right].$$

This coordinate transformation yields

$$\dot{\phi} \;=\; S^{-1}(-EKE^T)S\phi \,+\, S^{-1}d$$

or, equivalently,

$$\left[\begin{array}{c} \dot{\psi} \\ \dot{\bar{x}} \end{array}\right] = \left[\begin{array}{cc} -E_t^T E_t R K R^T & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} \psi \\ \bar{x} \end{array}\right] + \left[\begin{array}{c} E_t^T \\ (1/N)\mathbf{1}^T \end{array}\right] d.$$

The generalized performance output in new coordinates is given by

$$\eta = \left[\begin{array}{c} q^{1/2} H_s \\ -r^{1/2} L \end{array}\right] S\phi = \left[\begin{array}{cc} q^{1/2} H_s E_t (E_t^T E_t)^{-1} & 0 \\ -r^{1/2} E_t R K R^T & 0 \end{array}\right] \left[\begin{array}{c} \psi \\ \bar{x} \end{array}\right].$$

By removing the average-mode $\bar{x}$, we obtain a minimal realization of the closed-loop system containing only the state ψ,

$$\begin{array}{rcl} \dot{\psi} &=& -E_t^T E_t P_K \psi \,+\, E_t^T d \\ \eta &=& \left[\begin{array}{c} q^{1/2} H_s \Sigma \\ -r^{1/2} E_t P_K \end{array}\right] \psi, \end{array} \tag{2}$$

where

$$P_K \,:=\, R\,K\,R^T, \;\; \Sigma \,:=\, E_t\left(E_t^T\,E_t\right)^{-1}.$$

The H_2 norm of system (2) from d to η is determined by

$$J(K) \,=\, \mathrm{trace}\left((q\Sigma^T H_s^T H_s \Sigma + r P_K E_t^T E_t P_K)\Psi\right)$$

where Ψ is the solution of the Lyapunov equation

$$-E_t^T E_t P_K \Psi \,-\, \Psi P_K E_t^T E_t \,+\, E_t^T E_t \,=\, 0.$$

As shown in Proposition 1, $-E_t^T E_t P_K$ is a Hurwitz matrix if and only if $P_K > 0$. In this case, the unique solution of the Lyapunov equation is given by $\Psi = 0.5 P_K^{-1}$, thereby simplifying the expression for the H_2 norm

$$J(K) = 0.5\,\mathrm{trace}\left(q\,\Sigma^T H_s^T H_s \Sigma P_K^{-1} \,+\, r\,P_K E_t^T E_t\right).$$

Therefore, the design of the optimal diagonal feedback gain K amounts to solving the following optimization problem

$$\begin{array}{ll} \text{minimize} & 0.5\,\mathrm{trace}\left(q\,\Sigma^T H_s^T H_s \Sigma P_K^{-1} \,+\, r\,P_K E_t^T E_t\right) \\ \text{subject to} & K \text{ is a diagonal matrix.} \end{array} \tag{P}$$

III. Main results: stability and convexity

This section contains the main results of the paper, given by Propositions 1 and 4. We show that (P) is a convex problem on the set of stabilizing feedback gains. Therefore, any minimizer of (P) is a *global* minimizer [6], and it is also guaranteed to achieve closed-loop stability.

Proposition 1: The closed-loop system (2) is stable if and only if $P_K = RKR^T$ is a positive definite matrix. In this case, the eigenvalues of $-E_t^T E_t P_K$ are all negative real numbers.

Proof: Since $E_t^T E_t$ is positive definite and P_K is Hermitian, the result follows immediately from Proposition 2. ■

Proposition 2: [9, Theorem 7.6.3] The product $W_1 W_2$, of a positive definite matrix W_1 and a Hermitian matrix W_2, has the same number of positive, negative, and zero eigenvalues as W_2.

A sufficient condition for closed-loop stability is $K > 0$. This condition is also necessary for tree graphs since in that case $R = I$ and $P_K = K$. Noting that K is a diagonal matrix, we have the following result.

Proposition 3: For tree graphs, the closed-loop system (2) is stable if and only if $K > 0$, i.e., $\{k_n\}_{n=1}^{|\mathcal{E}|} > 0$.

We next establish the convexity of the optimization problem (P).

Proposition 4: $J(K)$ in (P) is a convex function of the stabilizing feedback gain K.

Proof: We first observe that the stabilizing feedback gains form a convex set. For two arbitrary stabilizing gains K_1 and K_2, we have

$$R(\theta K_1 + (1-\theta)K_2)R^T \,>\, 0, \quad \theta \in (0,1),$$

that is, the convex combination $\theta K_1 + (1-\theta)K_2$ is also stabilizing. Since $\mathrm{trace}\,(P_K E_t^T E_t)$ is linear (and thus convex) in K, it suffices to show convexity of the function $\mathrm{trace}\,(Q P_K^{-1})$, where we denote

$$Q \,=\, q\,\Sigma^T H_s^T H_s \Sigma.$$

To this end, we use the fact [6, Problem 3.18(a)] that $\mathrm{trace}\,(W^{-1})$ is a convex function of the positive definite matrix W. Thus, $\mathrm{trace}\,(P_K^{-1})$ is convex for $P_K > 0$. Since $Q > 0$ and the positive weighted sum preserves convexity [6, Section 3.2.1], it follows that $\mathrm{trace}\,(Q P_K^{-1})$ is a convex function of the stabilizing feedback gain K. ■

Since we are minimizing a convex objective function on a convex set, (P) is a convex optimization problem [6].

It is also instructive to show the positive semi-definiteness of the Hessian matrix to conclude the convexity of $J(K)$ [6]. Here, we give the formulas for gradient and Hessian of $J(K)$, with detailed derivations provided in Appendix A

$$\begin{aligned} \nabla J(K) \,&=\, 0.5\,\mathrm{diag}(R^T(r\,E_t^T E_t - P_K^{-1} Q P_K^{-1})R), \\ \nabla^2 J(K) \,&=\, 0.5\,(R^T P_K^{-1} Q P_K^{-1} R) \circ (R^T P_K^{-1} R), \end{aligned}$$

where $\mathrm{diag}(W)$ is the main diagonal of the matrix W, and $\circ$ is the entry-wise multiplication of two matrices. We can utilize descent methods [6], e.g., Newton's method, to compute the global solution of (P).

IV. Analytical solutions

In this section, by exploiting structure of path, star, circle, and complete graphs we provide analytical solutions to problem (P). The local and global performance measures are determined by $J_s(K)$ with respect to H_s for $s = l$ or $s = g$. Recall that $H_l = E^T$ and consequently

$$Q_l \,=\, q\,\Sigma^T E E^T \Sigma \,=\, q\,RR^T,$$

implying that the local performance measure is given by

$$J_l(K) \,=\, 0.5\,\mathrm{trace}\left(q\,RR^T P_K^{-1} \,+\, r\,P_K E_t^T E_t\right). \tag{LP}$$

On the other hand, $H_g = I - (1/N)\mathbf{1}\mathbf{1}^T$ in combination with $E_t^T \mathbf{1} = 0$ yields

$$Q_g \,=\, q\,\Sigma^T (I - (1/N)\mathbf{1}\mathbf{1}^T)^2 \Sigma \,=\, q\,(E_t^T E_t)^{-1},$$

implying that the global performance measure is given by

$$J_g(K) \,=\, 0.5\,\mathrm{trace}\left(q\,(P_K E_t^T E_t)^{-1} \,+\, r\,P_K E_t^T E_t\right). \tag{GP}$$

A. Tree

A tree is a connected graph with no cycles. In this case, $R = I$ and (LP) simplifies to

$$J_l(K) \,=\, 0.5\,\mathrm{trace}\,(q\,K^{-1} \,+\, r\,K E_t^T E_t).$$

By the definition of the incidence matrix, the diagonal entries of $E_t^T E_t$ are all equal to 2 yielding

$$J_l(K) \,=\, 0.5 \sum_{i=1}^{N-1} (q k_i^{-1} + 2 r k_i).$$

The unique optimal feedback gain is thus obtained when

$$q k_i^{-1} \,=\, 2 r k_i, \quad i \in \{1, \ldots, N-1\},$$

which yields a *constant* gain for all the $N-1$ edges

$$(k_l^*)_i \;=\; \sqrt{q/(2r)} \;=:\; C_k.$$

The optimal local performance measure is given by

$$J_l^* \;=\; \textstyle\sum_{i=1}^{N-1} 2r(k_l^*)_i \;=\; C_p(N-1),$$

where $C_p := 2rC_k = \sqrt{2qr}$.

For (GP), we have

$$\begin{aligned} J_g(K) &= 0.5\,\mathrm{trace}\,(q(KE_t^TE_t)^{-1} \,+\, rKE_t^TE_t) \\ &= 0.5\textstyle\sum_{i=1}^{N-1}\left(qk_i^{-1}\big((E_t^TE_t)^{-1}\big)_{ii} \,+\, 2rk_i\right). \end{aligned}$$

The optimal feedback gain is thus obtained when

$$qk_i^{-1}((E_tE_t^T)^{-1})_{ii} \;=\; 2rk_i$$

which yields

$$(k_g^*)_i \;=\; C_k\sqrt{\big((E_t^TE_t)^{-1}\big)_{ii}}, \quad i \in \{1,\ldots,N-1\}.$$

The optimal global performance is

$$J_g^* \;=\; C_p\textstyle\sum_{i=1}^{N-1}\sqrt{\big((E_t^TE_t)^{-1}\big)_{ii}}.$$

Thus, in contrast to (LP), the optimal feedback gain and the optimal global performance measure depend on the structure of the tree. We next consider two special cases where we can determine $(E_t^TE_t)^{-1}$ explicitly.

Path: In this case, the nth column of E_t has 1 and -1 as the nth and $(n+1)$th entries, respectively. For example, $N=4$

$$E_t \;=\; \left[\begin{array}{ccc} e_1 & e_2 & e_3 \end{array}\right] \;=\; \left[\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right].$$

Therefore

$$E_t^TE_t \;=\; T$$

where T is a symmetric Toeplitz matrix with the first row given by $[\;2\;\; -1\;\; 0\;\; \cdots\;\; 0\;] \in \mathbb{R}^{N-1}$. It can be shown [10] that the ijth entry of the symmetric matrix T^{-1} is given by

$$(T^{-1})_{ij} \;=\; i(N-j)/N \quad \text{for} \quad j \,\geq\, i. \tag{3}$$

It follows that

$$\begin{aligned} (k_g^*)_i &= C_k\sqrt{i(N-i)/N}, \\ J_g^* &= C_p\textstyle\sum_{i=1}^{N-1}\sqrt{i(N-i)/N}. \end{aligned}$$

For large N,

$$J_g^* \approx (\pi C_p/8)N\sqrt{N},$$

which follows from the calculation

$$\begin{aligned} \lim_{N\to\infty} J_g^*/(N\sqrt{N}) \;&=\; C_p\lim_{N\to\infty}\sum_{i=1}^{N-1}\sqrt{\frac{i}{N}-\frac{i^2}{N^2}}\frac{1}{N} \\ &=\; C_p\int_0^1\sqrt{\theta-\theta^2}\,\mathrm{d}\theta \;=\; \pi C_p/8. \end{aligned}$$

Star: In this case,

$$E_t^T \;=\; \left[\begin{array}{cc} \mathbf{1} & -I \end{array}\right]$$

and therefore

$$E_t^TE_t \;=\; I \,+\, \mathbf{1}\mathbf{1}^T \;=:\; M \in \mathbb{R}^{(N-1)\times(N-1)}.$$

It is readily verified that

$$M^{-1} \;=\; I \,-\, (1/N)\mathbf{1}\mathbf{1}^T. \tag{4}$$

Thus the diagonal entries of $(E_t^TE_t)^{-1}$ are all equal to $(N-1)/N$. The optimal feedback gain and the global performance measure are thus given by

$$\begin{aligned} (k_g^*)_i \;&=\; C_k\sqrt{(N-1)/N}, \\ J_g^* \;&=\; C_p(N-1)\sqrt{(N-1)/N}. \end{aligned}$$

B. Circle

Circle is an edge-transitive graph [5] and we use the result that the optimal solution for convex problems on edge-transitive graphs are constant [7], [8], that is, $k_i^* = k$ for all $i \in \mathcal{V}$. The (LP) measure thus simplifies to

$$\begin{aligned} J_l(k) &= 0.5\,\mathrm{trace}\left(qRR^T(kRR^T)^{-1} + rkRR^TE_t^TE_t\right) \\ &= 0.5\,qk^{-1}(N-1) \,+\, rkN, \end{aligned}$$

where we used

$$\mathrm{trace}\left(RR^TE_t^TE_t\right) \;=\; \mathrm{trace}\left(E^TE\right) \;=\; 2N.$$

Thus,

$$\begin{aligned} k_l^* \;&=\; C_k\sqrt{(N-1)/N}, \\ J_l^* \;&=\; C_p(N-1)\sqrt{(N-1)/N}. \end{aligned}$$

The incidence matrix for circle graph is a circulant matrix with the first column given by $[\;1\;\; -1\; 0\;\; \cdots\;\; 0\;]^T \in \mathbb{R}^{N-1}$; for example, $N=4$,

$$E \;=\; \left[\begin{array}{rrrr} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right].$$

It follows that $R \,=\, \left[\begin{array}{cc} I & -\mathbf{1} \end{array}\right]$ and thus

$$RR^T \;=\; I + \mathbf{1}\mathbf{1}^T \;=\; M.$$

Since the tree subgraph of the circle is the path, we have $E_t^TE_t = T$. Multiplying T^{-1} in (3) and M^{-1} in (4) yields the ith diagonal entry of $T^{-1}M^{-1}$

$$(T^{-1}M^{-1}) \;=\; i(N-i)/(2N).$$

Thus, we have

$$\begin{aligned} J_g(k) &= 0.5\,\mathrm{trace}\left(qk^{-1}T^{-1}M^{-1} \,+\, rkRR^TE_t^TE_t\right) \\ &= 0.5((q/k)\textstyle\sum_{i=1}^{N-1}(i(N-i))/(2N) \,+\, 2rkN) \\ &= q(N^2-1)/(24k) \,+\, rkN. \end{aligned}$$

It follows that

$$\begin{aligned} k_g^* \;&=\; C_k\sqrt{(N^2-1)/(12N)}, \\ J_g^* \;&=\; C_pN\sqrt{(N^2-1)/(12N)}. \end{aligned}$$

C. Complete graph

A complete graph has an edge between every pair of vertices and thus it has $N(N-1)/2$ edges. A complete graph is also edge-transitive [5] and with an identical procedure as done for the case of circle, for (LP) we have

$$k_l^* \;=\; C_k\sqrt{2/N}, \quad J_l^* \;=\; C_p(N(N-1)/2)\sqrt{2/N}.$$

On the other hand, for (GP) we have

$$\begin{aligned} J_g(k) &= 0.5\,\mathrm{trace}\left(q(kRR^TE_t^TE_t)^{-1} + rkRR^TE_t^TE_t\right) \\ &= \;0.5\,(N-1)(q/(kN) \,+\, rkN). \end{aligned}$$

where we used the fact that the eigenvalues of $RR^TE_t^TE_t$ are the nonzero eigenvalues of E^TE, which all equal to N [5]. Thus, we have

$$k_g^* \;=\; C_k\sqrt{2}/N, \qquad J_g^* \;=\; C_p(N(N-1)/2)(\sqrt{2}/N).$$

We summarize the formulas for the optimal feedback gains and the optimal performances (LP) and (GP) in Table I.

D. Asymptotic scaling with network size

We next consider how the optimal performance measures scale with the network size N. We also consider the asymptotic scaling of the optimal feedback gain $\{k_i^*\}_{i=1}^{|\mathcal{E}|}$, which serves as an indication of the control effort of the optimal design. The results for the graphs considered are summarized in Table II, where the optimal local and global performance measures are normalized by N. The normalized global performance measures are illustrated in Fig. 2.

Clearly, the star graph is the most favorable structure with respect to both the performance measures and the control effort. The path and circle have the same asymptotic scaling. It is also noteworthy that k^* for complete graph decays as the size of network increases.

In [3], it was established that using constant feedback gain for all edges of the circle, the global performance measure normalized by the formation size scales *linearly* with N. This result was derived under the assumption that the amount of control effort is formation-size-independent. Note that the scaling of J_g^*/N can be reduced to a square-root dependence of N at the expense of k_g^* also increasing as a square-root function of N. To obtain k_g^* that does not increase with N, we select $r = N$ which results into

$$J_g(k) \;=\; q(N^2-1)/(24k) \,+\, kN^2.$$

Therefore, the optimal feedback gain in this case is given by

$$k_g^* \;=\; \sqrt{q(N^2-1)/(24N^2)},$$

which becomes N-independent as the number of vehicles goes to infinity. On the other hand,

$$J_g^* \;=\; N^2\sqrt{q(N^2-1)/(24N^2)},$$

which means that J_g^*/N scales as a linear function of N, the result in agreement with [3].

It is also noteworthy that the optimal feedback gains for local performance measure are constant for tree, circle and complete graph. If we restrict gains to be constant for a general graph $\mathcal{G}$, then the optimal localized gain and the corresponding optimal local performance measure are determined by

$$k_l^* \;=\; C_k\sqrt{(N-1)/|\mathcal{E}|}, \quad J_l^* \;=\; C_p\sqrt{(N-1)|\mathcal{E}|},$$

where $|\mathcal{E}|$ denotes the number of edges. Therefore, the favorable graphs for local performance measure should have the minimum number of edges for a fixed number of nodes, that is, the *tree* graph. In that case, $|\mathcal{E}| = N-1$, and thus we recover the result

$$k_l^* \;=\; C_k, \quad J_l^* \;=\; C_p(N-1),$$

obtained in Section IV-A.

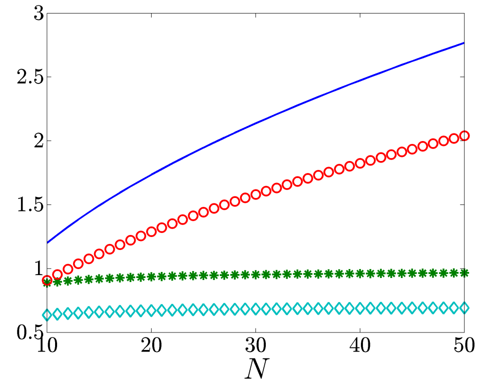


Fig. 2: Normalized global performance measure J_g^*/N with $C_p = 1$ for path $(-)$, circle $(\circ)$, star $(*)$ and complete graph $(\Diamond)$.

TABLE II: Comparison of the asymptotic scaling of optimal performance measures normalized by N, and the maximum value of feedback gains, for path, circle, star and complete graph.

	J_l^*/N	J_g^*/N	k_l^*	$\max(k_g^*)_i$
Path	$O(1)$	$O(\sqrt{N})$	$O(1)$	$O(\sqrt{N})$
Circle	$O(1)$	$O(\sqrt{N})$	$O(1)$	$O(\sqrt{N})$
Star	$O(1)$	$O(1)$	$O(1)$	$O(1)$
Complete	$O(\sqrt{N})$	$O(1)$	$O(\sqrt{1/N})$	$O(1/N)$

V. Concluding remarks

In this paper, we consider the design of optimal localized feedback gains for undirected connected graphs. We characterize the stabilizing feedback gains and demonstrate the convexity of the corresponding optimal control problem. Furthermore, we obtain explicit formulas for the optimal localized gains and determine analytically the asymptotic scaling of the performance measures with the network size for several graphs.

In this work, we focus on undirected graphs for which the relative feedback gains on the edges are symmetric, that is, $k_{ij} = k_{ji}$. One direction of future research is to consider the directed graphs with nonsymmetric feedback gains. In

TABLE I: Optimal gains and performances for the path, star, circle, and complete graph.

	J_l^*/C_p	k_l^*/C_k	J_g^*/C_p	$(k_g^*)_i/C_k$
Path	$N-1$	1	$(\pi/8)N\sqrt{N}$	$\sqrt{(i(N-i))/N}$
Star	$N-1$	1	$(N-1)\sqrt{(N-1)/N}$	$\sqrt{(N-1)/N}$
Circle	$(N-1)\sqrt{(N-1)/N}$	$\sqrt{(N-1)/N}$	$N\sqrt{(N^2-1)/(12N)}$	$\sqrt{(N^2-1)/(12N)}$
Complete	$(N(N-1)/2)\sqrt{2/N}$	$\sqrt{2/N}$	$(N(N-1)/2)(\sqrt{2}/N)$	$\sqrt{2}/N$

this case, the optimal control problem may become nonconvex. However, using the numerical schemes developed in [11], it is possible to obtain at least a local minimizer. We intend to examine this nonsymmetric case and compare the performances for both the symmetric and nonsymmetric designs in our future work.

APPENDIX

A. Gradient and Hessian of $J(K)$

Proposition 5: The Hessian

$$\nabla^2 J(K) \;=\; 0.5\,(R^T P_K^{-1} Q P_K^{-1} R) \circ (R^T P_K^{-1} R).$$

is a positive semi-definite matrix.

Proof: Since $\operatorname{trace}(r P_K E_t^T E_t)$ is linear in K, we have

$$\nabla^2 J_Q(K) \;=\; \nabla^2 J(K),$$

where

$$J_Q(K) \;:=\; 0.5\operatorname{trace}\left(Q P_K^{-1}\right).$$

We calculate

$$\begin{aligned} J_Q(K+\tilde{K}) \;&=\; 0.5\operatorname{trace}(Q(P_K + R\tilde{K}R^T)^{-1}) \\ &=\; 0.5\operatorname{trace}(Q(I + P_K^{-1}R\tilde{K}R^T)^{-1}P_K^{-1}) \end{aligned}$$

where diagonal matrix $\tilde{K}$ is the variation around K. For small $\tilde{K}$ such that

$$\|P_K^{-1} R\tilde{K}R^T\| < 1,$$

we expand the inverse

$$(I + P_K^{-1}R\tilde{K}R^T)^{-1} \approx I - P_K^{-1}R\tilde{K}R^T + (P_K^{-1}R\tilde{K}R^T)^2,$$

where we drop higher order terms in $\tilde{K}$. Therefore, the quadratic term of $\tilde{K}$ in $J_Q(K+\tilde{K})$ is given by

$$\begin{aligned} q(\tilde{K}) \;&=\; 0.5\operatorname{trace}(Q(P_K^{-1}R\tilde{K}R^T)^2 P_K^{-1}) \\ &=\; 0.5\operatorname{trace}(\tilde{K}R^T P_K^{-1} Q P_K^{-1} R\tilde{K}R^T P_K^{-1} R). \end{aligned}$$

Using Proposition 6, we have

$$q(\tilde{K}) \;=\; 0.5\,\tilde{k}^T\left((R^T P_K^{-1} Q P_K^{-1} R) \circ (R^T P_K^{-1} R)\right)\tilde{k},$$

where $\tilde{k} = \operatorname{diag}(\tilde{K})$ is the main diagonal of $\tilde{K}$. Since the entry-wise multiplication of two positive semi-definite matrices is also positive semi-definite [12, Theorem 5.2.1], we have

$$\nabla^2 J(K) \;\geq\; 0$$

and thus $J(K)$ is a convex function [6]. The gradient of $J(K)$ is readily obtained

$$\nabla J(K) \;=\; 0.5\operatorname{diag}(R^T(rE_t^T E_t - P_K^{-1} Q P_K^{-1})R),$$

by noting that the linear term of $\tilde{K}$ in $J_Q(K+\tilde{K})$ is given by

$$l(\tilde{K}) \;=\; -0.5\operatorname{trace}(R^T P_K^{-1} Q P_K^{-1} R\tilde{K}).$$

■

Proposition 6: [12, Lemma 5.1.5] For $W_1, W_2 \in \mathbb{R}^{m\times n}$, and $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we have

$$v_1^T(W_1 \circ W_2)v_2 \;=\; \operatorname{trace}(V_1 W_1 V_2 W_2),$$

where $\circ$ denotes the entry-wise multiplication of two matrices; and $V_i = \operatorname{diag}(v_i)$ for $i = 1, 2$ is a diagonal matrix with the main diagonal determined by v_i.

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