On the optimal localized feedback design for vehicular platoons

Fu Lin, Makan Fardad, and Mihailo R. Jovanović

***Abstract*— We consider the design of optimal structured feedback gains for vehicular platoons. We revisit the mistuning design problem proposed by Barooah *et al.*, where the platoon is modeled as a diffusion on a spatial lattice, and a search is performed for structured perturbations of the nominal dynamics that improve the stability properties in a favorable way. We pose the mistuning problem in the structured H_2 optimal control framework, where the size of the mistuning feedback gain is kept small by considering an expensive control regime. The coupled matrix equations that result from optimality conditions are conveniently decoupled via the application of perturbation analysis, which yields the unique structured optimal mistuning gain. We then consider less expensive control regimes and employ Newton's method to solve the optimal control problem, while using the solution obtained from mistuning to initialize the iterative homotopy-based scheme. We also consider the issue of scaling, with respect to platoon size, of global and local performance measures in the optimally-controlled platoon.**

***Index Terms*— Large-scale platoons, mistuning design, optimal control, perturbation analysis, homotopy, Newton's method.**

I. Introduction

The control of vehicular platoons has attracted considerable attention since the mid sixties [1]–[8]. This problem is emblematic of a wide range of technologically relevant applications including the control of UAVs, swarms of autonomous robotic agents, and satellite constellations. The vehicular platoon consists of a number of vehicles moving along a straight path. The simplest control objective for vehicles modeled as double integrators is to maintain a desired cruising velocity and to keep a pre-specified constant distance between neighboring vehicles.

Recent work in this area has focused on fundamental performance limitations in the design of large-scale platoons [5]–[8]. In [5], it was shown that stabilizability and detectability of LQR formulations based on penalizing relative position errors deteriorate as functions of a finite platoon size. In [6], it was shown that convergence of merge and split maneuvers can have poor scaling properties (with system size) even upon inclusion of absolute position errors in cost functionals.

The motivation for the current study comes from two recent papers [7] and [8]. In [7], fundamental performance limits of spatially invariant consensus and vehicular formation control problems in the presence of localized feedback were addressed. In particular, it was shown that spatially uniform local feedback is not capable of maintaining coherence in large-scale platoons. This was done by exhibiting linear scaling, with the number of vehicles, of the *normalized* H_2 norm from disturbances to an appropriately defined macroscopic performance measure, where normalization is done with respect to the platoon size. For platoons on a regular one-dimensional lattice, it was shown in [8] that the trends in the rate of decay of the least stable closed-loop eigenvalue (with the number of vehicles) can be improved in a favorable way by introducing a small amount of 'mistuning' in spatially uniform feedback gains. This method was based on first modeling a large platoon as a diffusive system, and then designing a small-in-norm perturbation profile that destroys the spatial symmetry and renders the system more stable. Numerical simulations were also used to show that the spatially-varying mistuned feedback gains have beneficial influence on the closed-loop H_∞ norm.

In this paper, we pose the mistuning problem in the structured H_2 optimal control framework [9], where the size of the mistuning feedback gain is kept small by considering an expensive control regime. Our approach differs from the original work of [8] in that the mistuning profile is found by optimizing a performance index rather than performing spectral analysis. The coupled matrix equations that result from optimality conditions are conveniently decoupled via the application of perturbation analysis, which yields an explicit analytical expression for the structured optimal mistuning gain. We then consider less expensive control regimes and employ Newton's method to solve the optimal control problem, while using the solution obtained from mistuning to initialize the iterative homotopy-based scheme.

Finally, we examine how the performance of the optimally-controlled platoon scales with the number of vehicles. We consider both macroscopic and microscopic performance measures based, respectively, on whether the objective is to minimize the absolute position error of every vehicle or the relative position error between neighboring vehicles. In particular, using a macroscopic performance measure we demonstrate that an optimally designed mistuning can substantially improve coherence of the large-scale formation.

II. Problem formulation

We consider a system of N identical unit mass vehicles moving in a line. As illustrated in Fig. 1, fictitious lead and follow vehicles, respectively indexed by 0 and $N+1$, are added to the formation. These two vehicles are assumed to move along their absolute desired trajectories at all time and they are not considered to belong to the platoon. The control objective is to keep neighboring vehicles at a pre-specified constant distance from each other. For simplicity, each vehicle is represented by its kinematic model,

$$\dot{x}_n \;=\; d_n + u_n, \;\; n \;=\; \{1,\ldots,N\},$$

where x_n denotes the difference between the position of the nth vehicle and its desired absolute position in a regular grid, u_n is the control applied on the nth vehicle, and d_n is the disturbance acting on the nth vehicle. Our goal is to quantify the influence of disturbances on both local and global performance measures for formations with static

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feedback controllers and nearest neighbor interactions. In all of our developments, we assume that all vehicles are equipped with ranging devices that allow them to measure relative distances with their immediate neighbors. Additionally, vehicles indexed by 1 and N are equipped with a GPS device which gives them access to their global positions at every time instant.

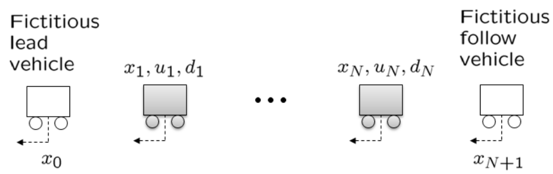


Fig. 1. A platoon of vehicles.

A. Spatially uniform controller

We first consider a spatially uniform static controller with nearest neighbor interactions,

$$u_n \;=\; -(x_n - x_{n-1}) \;-\; (x_n - x_{n+1}), \quad n \;=\; 1, \ldots, N.$$

Clearly, the control action for each vehicle is computed using relative position errors with respect to the neighboring vehicles; since lead and follow fictitious vehicles are constrained to move along their absolute desired trajectories, this means that the vehicles indexed by 1 and N also have access to their absolute positions. In matrix form, we have

$$u \;=\; -Kx \;=\; -Tx,$$

where x and u denote the state and control vectors, e.g. $x = [\, x_1 \;\; \cdots \;\; x_N \,]^T$, and T is an $N \times N$ symmetric Toeplitz matrix with the first row given by $[\, 2 \;\; -1 \;\; 0 \;\; \cdots \;\; 0 \,] \in \mathbb{R}^N$. Thus, the closed-loop system is determined by

$$\begin{aligned} \dot{x} &\;=\; -T\,x \,+\, d, \\ z &\;=\; \left[\begin{array}{c} z_1 \\ z_2 \end{array} \right] \;=\; \left[\begin{array}{c} Q^{1/2} \\ -R^{1/2}\,T \end{array} \right] x, \end{aligned} \qquad \text{(SU)}$$

where d is a vector of disturbances, and z is a vector of performance variables that encompass both the penalty Q on the state, and the penalty R on the control. In what follows, the control weight is given by a scaled identity, $R = rI$ with $r > 0$. On the other hand, we will consider two state performance measures

- Macroscopic (global), $Q_g = I$;
- Microscopic (local), $Q_l = T$.

These state weights induce two performance measures of interest determined by the H_2 norm of the closed-loop system (from d to z_1) normalized with the number of vehicles,

$$\Pi_s(N) \;=\; (1/N) \|H\|_2^2, \quad s \,=\, g \;\;\text{or}\;\; s \,=\, l. \qquad (\Pi)$$

B. Mistuning

With the choice of spatially uniform static feedback gain in Section II-A we have effectively closed the loop with the nominal 'diffusive' dynamics; this form of control compares the position of each vehicle with the average of positions of its immediate neighbors and it is commonly encountered in standard consensus algorithms. Following [8], we next augment the 'diffusive' feedback with another control input

$$u \;=\; -Tx \,+\, v.$$

The role of v is to mistune the spatially uniform feedback gains and our objective is to examine how mistuning influences the scaling of the functions Π_g and Π_l with the number of vehicles. We will determine the mistuning profile that leads to minimization of the H_2 norm with respect to the previously defined performance measures under the constraint that v_n only utilizes relative position information. In particular,

$$v_n = -f_n\,(x_n - x_{n-1}) - b_n\,(x_n - x_{n+1})\,,\; n \,=\, \{1, \ldots, N\},$$

where f_n and b_n, respectively, denote the forward and backward 'mistuned' feedback gains which are to be determined using structured H_2 optimal control framework [9].

We consider the following control problem

$$\begin{aligned} \dot{x} &\;=\; A\,x \,+\, B_1\,d \,+\, B_2\,v, \\ z &\;=\; C_1\,x \,+\, D\,v, \\ y &\;=\; C_2\,x, \qquad u \;=\; -F\,y, \end{aligned}$$

where d denotes a mutually uncorrelated white stochastic process,

$$\begin{array}{ll} A \;=\; -T, & B_1 \;=\; B_2 \;=\; I, \\ C_1 \;=\; \left[\; Q^{1/2} \quad 0 \;\right]^T, & D \;=\; \left[\; 0 \quad r^{1/2} I \;\right]^T, \\ C_2 \;=\; \left[\begin{array}{c} C_f \\ C_f^T \end{array} \right], & F \;=\; [\; F_f \quad F_b \;], \end{array} \qquad \text{(VP)}$$

and

$$C_f \;\sim\; \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array} \right].$$

The sign $\sim$ means that the matrix on the right only exemplifies the structure and gives the element values of the actual matrix; in general the actual matrix can have much larger dimension. The interpretation of the output matrices is that $C_f\,x$ gives the vector of relative position errors between every vehicle and the one in front of it, and $C_f^T\,x$ gives the vector of relative position errors between every vehicle and the one behind it. Finally,

$$F_f \;\sim\; \left[\begin{array}{cccc} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & f_4 \end{array} \right], \quad F_b \;\sim\; \left[\begin{array}{cccc} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{array} \right],$$

and the elements of the real vectors $f = [\, f_1 \;\; \cdots \;\; f_N \,]^T$, $b = [\, b_1 \;\; \cdots \;\; b_N \,]^T$ can be interpreted as the 'weight' that each vehicle puts on the relative position error between itself and the vehicles in front it and behind it, respectively.

Upon closing the loop, the above problem can equivalently be written as

$$\begin{aligned} \dot{x} &\;=\; (A - FC_2)\,x \,+\, d, \\ z &\;=\; \left[\begin{array}{c} Q^{1/2} \\ -r^{1/2} F C_2 \end{array} \right] x, \end{aligned} \qquad \text{(H2)}$$

and thus the amplitude of z encapsulates both the amplitude of the state and that of the control input v.

The motivation for considering structured control problem (H2) comes from a recent study [8] where a continuum model that approximates a large platoon by a partial differential equation (PDE) was introduced. A 'mistuning profile' εF was then selected to maximally increase the temporal decay rate of the PDE, subject to a norm bound on F. The authors achieve this by performing a perturbation analysis on the spectrum of the operator $A_m(F, \varepsilon) = A - \varepsilon F C_2$ for small ε and by choosing F such that the rightmost part of the spectrum of A_m is pushed as far left into the complex plane as possible. Our approach to this problem is to minimize the H_2 norm of the closed-loop system (H2). This contrasts with the work of [8] in that the feedback gain F will be found by optimizing a performance index rather than performing a spectral analysis.

It can be shown that necessary conditions for F to be the optimal structured gain are given by [9]

$$\begin{aligned}(A - FC_2)^T P + P(A - FC_2) &= -\left(Q + rC_2^T F^T F C_2\right) \\ (A - FC_2) L + L (A - FC_2)^T &= -I \qquad \text{(NC)} \\ r\left(FC_2 L C_2^T\right) \circ I_{\mathcal{S}} &= \left(PLC_2^T\right) \circ I_{\mathcal{S}},\end{aligned}$$

where $\circ$ denotes the element-wise multiplication of matrices and $I_{\mathcal{S}} = [\,I \;\; I\,]$. For example, $X \circ I_{\mathcal{S}}$ is a matrix whose blocks are diagonal matrices determined by $X \circ I_{\mathcal{S}} = [\,X_1 \circ I \;\; X_2 \circ I\,]$, where X is partitioned conformably with the static feedback gain $F = [\,F_f \;\; F_b\,]$.

The conditions (NC) consist of two Lyapunov equations in P and L that are *coupled* together by the final equation. These equations can have multiple solutions, each of which is a stationary point of the objective function. In general, it is not known how many local minima exist or how to find them. This difficulty persists even in the unstructured problems, as pointed out by [10]. Recent work of the authors [9] has demonstrated how these issues can be circumvented in the framework of structured expensive control with $r = 1/\varepsilon$, $0 < \varepsilon \ll 1$.

The rest of the paper is organized as follows. In Section III, we utilize perturbation analysis to determine a small-amplitude mistuning correction to the nominal diffusive dynamics. In Section IV, we use a homotopy-based Newton's method to determine the solution to equations (NC) for smaller values of r; the Newton's iterations are initialized by a mistuning profile determined in Section III. This illustrates how the mistuning profile changes as one departs from the expensive control regime. In Section V we discuss how optimal localized control design influences microscopic and macroscopic performance measures, $\Pi_l(N)$ and $\Pi_g(N)$.

III. Optimal mistuning: expensive control

In this section, we consider a simpler problem in which $R = (1/\varepsilon)\, I$, with $0 < \varepsilon \ll 1$; we will henceforth refer to this as *expensive optimal control*. Then, by representing P, L, and F as

$$P = \sum_{n=0}^{\infty} \varepsilon^n P^{(n)}, \quad L = \sum_{n=0}^{\infty} \varepsilon^n L^{(n)}, \quad F = \sum_{n=1}^{\infty} \varepsilon^n F^{(n)},$$

substituting in (NC), and employing perturbation analysis, we obtain the set of *conveniently coupled* equations given by (EXP). Note that these equations are only coupled in one direction, in the sense that for any $n \geq 1$ the $O(\varepsilon^n)$ equations depend only on the solutions of the $O(\varepsilon^{n-1})$ equations and are not coupled among themselves. Thus the perturbation expansion terms can be readily computed up to any order.

The matrix F found by this procedure is the *unique optimal* (in the sense of perturbations) solution of the control problem (H2) with $r = 1/\varepsilon$. This is due to the fact that the equations (EXP), under the assumption of convergence, give a unique matrix $F = \sum_{n=1}^{\infty} \varepsilon^n F^{(n)}$. To compute the optimal mistuning profiles for larger values of ε, we use F from perturbation analysis to initialize homotopy-based Newton's method, as discussed in Section IV.

We next determine the optimal mistuning profile up to first order in ε for both macroscopic and microscopic performance measures.

A. Optimal mistuning profile for $Q_g = I$

Proposition 1: Up to first order in ε, the optimal H_2 feedback gains $F_f^{(1)} = \operatorname{diag}\{f_n^{(1)}\}$ and $F_b^{(1)} = \operatorname{diag}\{b_n^{(1)}\}$ in the control problem (H2)-(VP) with $Q = I$, $r = 1/\varepsilon$, $0 < \varepsilon \ll 1$, are given by

$$\begin{aligned} f_n^{(1)} &= \frac{n(n-N-1)(4n(N+1) - N(2N+7) + 1)}{12\,(N^2-1)}, \\ b_n^{(1)} &= -\frac{n(n-N-1)(4n(N+1) - N(2N+1) - 5)}{12\,(N^2-1)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} f_n^{(1)} + b_n^{(1)} &= 0.5\, n(N+1-n)/(N+1), \\ f_n^{(1)} &= b_{N+1-n}^{(1)}, \quad n = 1, \ldots, N. \end{aligned}$$

Proof: From $O(\epsilon)$ equations of (EXP) and (VP) we have

$$\begin{aligned} -T^T P^{(0)} - P^{(0)} T &= -I, \\ -T\, L^{(0)} - L^{(0)} T^T &= -I. \end{aligned}$$

Using the symmetry of T, we obtain

$$P^{(0)} = L^{(0)} = 0.5\,\Gamma,$$

where $\Gamma = T^{-1}$ is a symmetric matrix with entries determined by

$$\gamma_{ij} = i(N+1-j)/(N+1), \quad j \geq i. \qquad (1)$$

Substitution of $P^{(0)}$ and $L^{(0)}$ in the last $O(\varepsilon)$ equation of (EXP) yields

$$0.5 (F^{(1)} C_2 \Gamma C_2^T) \circ I_{\mathcal{S}} = 0.25 (\Gamma^2 C_2^T) \circ I_{\mathcal{S}}. \qquad (2)$$

The left hand side of (2) can be simplified to obtain

$$\begin{aligned} &(F^{(1)} C_2 L^{(0)} C_2^T) \circ I_{\mathcal{S}} \\ &\quad = 0.5 \left[F_f^{(1)} C_f \Gamma C_f^T + F_b^{(1)} C_f^T \Gamma C_f^T, \right. \\ &\qquad\quad \left. F_f^{(1)} C_f \Gamma C_f + F_b^{(1)} C_f^T \Gamma C_f \right] \circ [\,I \;\; I\,] \\ &\quad = 0.5 \left[F_f^{(1)} (C_f \Gamma C_f^T) \circ I + F_b^{(1)} (C_f^T \Gamma C_f^T) \circ I, \right. \\ &\qquad\quad \left. F_f^{(1)} (C_f \Gamma C_f) \circ I + F_b^{(1)} (C_f^T \Gamma C_f) \circ I \right], \end{aligned} \qquad \text{(LHS)}$$

where in the first equality we use (VP) to substitute for C_2 and $F^{(1)}$, and in the second equality we use the fact that for two matrices X and Y, with X diagonal,

$$(XY) \circ I = X\,(Y \circ I).$$

$$
\begin{aligned}
O(1): &\quad F^{(0)} \;=\; 0 \\
O(\varepsilon): &\begin{cases} A^T P^{(0)} \,+\, P^{(0)} A \;=\; -Q \\ A L^{(0)} \,+\, L^{(0)} A^T \;=\; -B_1 B_1^T \\ (F^{(1)} C_2 L^{(0)} C_2^T) \circ I_{\mathcal{S}} \;=\; (B_2^T P^{(0)} L^{(0)} C_2^T) \circ I_{\mathcal{S}} \end{cases} \\
O(\varepsilon^2): &\begin{cases} A^T P^{(1)} \,+\, P^{(1)} A \;=\; (B_2 F^{(1)} C_2)^T P^{(0)} \,+\, P^{(0)} (B_2 F^{(1)} C_2) \,-\, C_2^T F^{(1)T} F^{(1)} C_2 \\ A L^{(1)} \,+\, L^{(1)} A^T \;=\; (B_2 F^{(1)} C_2) L^{(0)} \,+\, L^{(0)} (B_2 F^{(1)} C_2)^T \\ (F^{(2)} C_2 L^{(0)} C_2^T) \circ I_{\mathcal{S}} \;=\; (B_2^T P^{(0)} L^{(1)} C_2^T \,+\, B_2^T P^{(1)} L^{(0)} C_2^T \,-\, F^{(1)} C_2 L^{(1)} C_2^T) \circ I_{\mathcal{S}} \end{cases} \\
\vdots &\qquad\qquad\qquad \vdots
\end{aligned}
\tag{EXP}
$$

Simplifying the right hand side of (2) we have

$$
\begin{aligned}
&(P^{(0)} L^{(0)} C_2^T) \circ I_{\mathcal{S}} \\
&\;=\; 0.25 \begin{bmatrix} \Gamma^2 C_f^T & \Gamma^2 C_f \end{bmatrix} \circ \begin{bmatrix} I & I \end{bmatrix} \\
&\;=\; 0.25 \begin{bmatrix} (\Gamma^2 C_f^T) \circ I & (\Gamma^2 C_f) \circ I \end{bmatrix}
\end{aligned}
\tag{RHS}
$$

We now equate (LHS) and (RHS) and solve for the unknown elements $f_n^{(1)}$ and $b_n^{(1)}$ of the diagonal matrices $F_f^{(1)}$ and $F_b^{(1)}$,

$$
\begin{aligned}
F^{(1)} &\;=\; \begin{bmatrix} F_f^{(1)} & F_b^{(1)} \end{bmatrix} \\
&\;=\; \begin{bmatrix} \operatorname{diag}\{f_n^{(1)}\} & \operatorname{diag}\{b_n^{(1)}\} \end{bmatrix}.
\end{aligned}
$$

Exploiting the diagonal structure of $F_f^{(1)}$, $F_b^{(1)}$, and $(\,\cdot\,) \circ I$, the matrix equation collapses to the set of scalar equations

$$
\begin{gathered}
[C_f \Gamma C_f^T]_{nn}\, f_n^{(1)} + [C_f^T \Gamma C_f^T]_{nn}\, b_n^{(1)} \;=\; 0.5\, [\Gamma^2 C_f^T]_{nn} \\
[C_f \Gamma C_f]_{nn}\, f_n^{(1)} + [C_f^T \Gamma C_f]_{nn}\, b_n^{(1)} \;=\; 0.5\, [\Gamma^2 C_f]_{nn} \\
\text{for } n \;=\; 1, \ldots, N.
\end{gathered}
$$

Solving this 2-by-2 system of equations for $f_n^{(1)}$ and $b_n^{(1)}$ yields the expressions for $f_n^{(1)}$ and $b_n^{(1)}$ in Proposition 1. ■

B. Optimal mistuning profile for $Q_l = T$

Proposition 2: Up to first order in ε, the optimal H_2 feedback gains $F_f^{(1)} = \operatorname{diag}\{f_n^{(1)}\}$ and $F_b^{(1)} = \operatorname{diag}\{b_n^{(1)}\}$ in control problem (H2)-(VP) with $Q = T$, $r = 1/\varepsilon$, $0 < \varepsilon \ll 1$, are given by

$$
f_n^{(1)} \;=\; 0.5(N-n)/(N-1), \;\; b_n^{(1)} \;=\; 0.5(n-1)/(N-1).
$$

Furthermore,

$$
f_n^{(1)} \,+\, b_n^{(1)} \;=\; 0.5, \;\; f_n^{(1)} \;=\; b_{N+1-n}^{(1)}, \;\; n \;=\; 1, \ldots, N.
$$

Proof: From $O(\epsilon)$ equations of (EXP) and (VP) we have

$$
\begin{aligned}
-T^T P^{(0)} \,-\, P^{(0)} T &\;=\; -T, \\
-T\, L^{(0)} \,-\, L^{(0)} T^T &\;=\; -I.
\end{aligned}
$$

Using the symmetry of T, we obtain

$$
P^{(0)} \;=\; 0.5\, I, \quad L^{(0)} \;=\; 0.5\, \Gamma.
$$

Following a procedure similar to that explained above for the case with $Q_l = I$ and performing some algebra yields the expressions for $f_n^{(1)}$ and $b_n^{(1)}$ in Proposition 2. ■

C. Optimal mistuning profile for $Q = \alpha I + \beta T$

From the $O(\varepsilon)$ equations in (EXP) we conclude that the mapping from Q to $F^{(1)}$ is linear. Specifically, $F^{(1)}$ is a linear function of $P^{(0)}$, and $P^{(0)}$ is a linear function of Q. Thus, the mistuning profiles for $Q = \alpha I + \beta T$ are given by

$$
f_n^{(1)} \;=\; \alpha\, f_{g,n}^{(1)} + \beta\, f_{l,n}^{(1)}, \quad b_n^{(1)} \;=\; \alpha\, b_{g,n}^{(1)} + \beta\, b_{l,n}^{(1)}
$$

where $f_{g,n}^{(1)}$ and $b_{g,n}^{(1)}$ are the mistuning profiles obtained with $Q_g = I$, whereas $f_{l,n}^{(1)}$ and $b_{l,n}^{(1)}$ are the mistuning profiles obtained with $Q_l = T$.

D. Optimal mistuning profile with assumptions made in [8]

We next compare the mistuning profiles described above with the optimal H_2 mistuning profile arising from the assumptions made in [8]. To this end, we introduce the following change of variables

$$
S \;=\; 0.5\,(F_f + F_b), \quad M \;=\; 0.5\,(F_f - F_b),
$$

or equivalently,

$$
F_f \;=\; S + M, \quad F_b \;=\; S - M.
$$

In the mistuning framework of [8], the matrix S is set to zero, and the diagonal elements $\{\mu_n\}$ of M determine the mistuning profile. Taking $S = 0$ implies that

$$
F_f \;=\; M, \quad F_b \;=\; -M \;=\; F_f.
$$

Clearly, this provides fewer degrees of freedom in design, since F_f and F_b cannot be selected independently of each other. Furthermore, the results summarized in Propositions 1 and 2 imply that optimal feedback gains F_b and F_f fail to satisfy $F_b = -F_f$.

Perturbation analysis of expensive control can now be used to obtain the optimal mistuning profile, $M^{(1)} = \operatorname{diag}\{\mu_n^{(1)}\}$, arising from the assumptions made in [8]. The result summarized in Proposition 3 can be proved in a similar manner to Propositions 1 and 2.

Proposition 3: Up to a first order in ε, the optimal H_2 feedback gains $M^{(1)} = F_f^{(1)} = \operatorname{diag}\{\mu_n^{(1)}\}$, $F_b^{(1)} = -F_f^{(1)}$, in control problem (H2)-(VP) with $r = 1/\varepsilon$, $0 < \varepsilon \ll 1$, are given by

$$
\mu_n^{(1)} \;=\; \begin{cases} \dfrac{n(N+1-2n)(N+1-n)}{6(N-1)} & Q \;=\; I \\[2ex] \dfrac{N+1-2n}{4(N-1)} & Q \;=\; T. \end{cases}
$$

Comparison of the $\mu_n^{(1)}$ obtained above with $f_n^{(1)}$ found in Propositions 1 and 2 reveals that $\mu_n^{(1)} \approx f_n^{(1)}$ when $Q = I$. However, when $Q = T$ then $\mu_n^{(1)}$ and $f_n^{(1)}$ differ significantly.

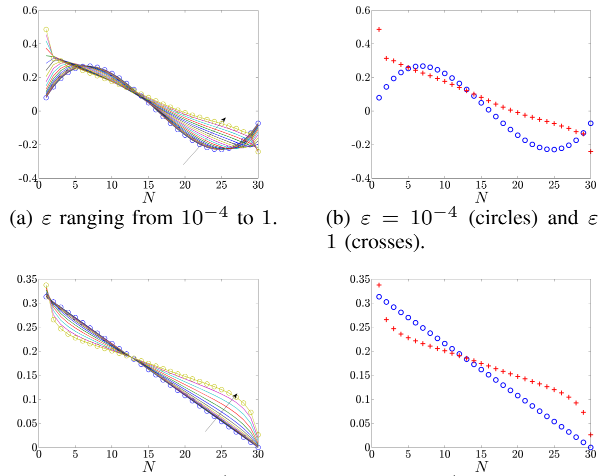
IV. Optimal mistuning: non-expensive control

In this section, we consider the optimal mistuning profile when the level of control expensiveness is reduced. To this end, we use recently developed Newton's method for the optimal design of static structured feedback gains [9]. Newton's method is an iterative descent algorithm to solve the optimization problem. Specifically, given an initial structured feedback gain $F^0 \in \mathcal{S}$, Newton's method generates a minimizing sequence $\{F^i \in \mathcal{S}\}$

$$F^{i+1} \;=\; F^i \,+\, s^i \tilde{F}^i_{nt},$$

where $\tilde{F}^i_{nt}$ is the Newton direction, s^i is the step-size, and $\mathcal{S}$ specifies the desired structure of the feedback gain. For very small values of ε, we initialize Newton's method using the optimal mistuning profile determined in Section III. We then increase ε and use the optimal F from Newton's method at previous ε to initialize the next round of Newton iterations. We continue increasing the value of ε until the control penalty $R = rI$ of the desired control objective is recovered.

Figure 2 illustrates the changes in the normalized optimal mistuning forward gains, $\{f_f/\|f_f\|;\; f_f \;=\; \mathrm{diag}\,\{F_f\}\}$, obtained using macroscopic ($Q_g \;=\; I$) and microscopic ($Q_l \;=\; T$) performance measures with $R \;=\; (1/\varepsilon)I$ and $N \;=\; 30$. The values of ε in homotopy-based method are determined by 20 logarithmically spaced points between 10^{-4} and 1. The mistuning profile changes continuously as the level of expensiveness of the control problem is reduced. Note that the feedback gains change gradually from an almost sinusoidal shape at $\varepsilon = 10^{-4}$ into a piecewise linear shape at $\varepsilon = 1$. The optimal forward and backward feedback gains in both cases also satisfy central symmetry property stated in Propositions 1 and 2, that is, $f_n = b_{N+1-n},\; n = \{1,\ldots,N\}$. It is noteworthy that optimal forward gains at $\varepsilon = 1$ monotonically decrease as one moves from the first vehicle to the last vehicle. Furthermore, the forward feedback gain at the first vehicle assumes much larger values compared to feedback gains of other vehicles.



(a) ε ranging from 10^{-4} to 1. (b) $\varepsilon \;=\; 10^{-4}$ (circles) and $\varepsilon \;=\; 1$ (crosses).

(c) ε ranging from 10^{-4} to 1. (d) $\varepsilon \;=\; 10^{-4}$ (circles) and $\varepsilon \;=\; 1$ (crosses).

Fig. 2. Normalized optimal forward mistuning, $f_f/\|f_f\|$, for a platoon with $N = 30$ vehicles and (a,b) $Q_g = I$; (c,d) $Q_l = T$.

V. Performance vs. platoon size

In this section, we study how optimal mistuning profiles with $R \;=\; I$ influence the scaling of macroscopic and microscopic performance measures as a function of the platoon size. For a given mistuning controller, we examine the improvement (or degradation) of the performances in both the local and global measures relative to the spatially uniform controller of Section II-A. In what follows, the optimal feedback gains of Section IV obtained with $Q_g = I$ and $Q_l = T$ are denoted by F_g and F_l, respectively.

A. Performance of spatially uniform controller

We next examine influence of platoon size on the performance of spatially uniform controller described in Section II-A. The H_2 norm of system (SU) from d to z is determined by the solution to the following Lyapunov equation,

$$(-T)^T P \,+\, P(-T) \;=\; -(Q \,+\, rT^T T),$$

where P denotes the closed-loop observability Gramian. Using the fact that $T = T^T$ and [11, Lemma 1] we have

$$\Pi_s \;\;=\;\; (0.5/N)\,\mathrm{trace}\left(T^{-1} Q_s\right).$$

Now, the fact that $\Gamma \;=\; T^{-1}$ is a symmetric matrix with entries determined by (1) can be used to show affine scaling with N of the macroscopic performance measure Π_g

$$\begin{aligned}\Pi_g(N) \;&=\; (0.5/N)\,\mathrm{trace}\,(\Gamma)\\ &=\; \frac{1}{2N}\sum_{n=1}^{N} n \,-\, \frac{1}{2N(N+1)}\sum_{n=1}^{N} n^2\\ &=\; (N+2)/12.\end{aligned}$$

On the other hand, for the microscopic performance measure, $Q_l \;=\; T$, we have the following platoon-size independent value of Π_l

$$\Pi_l \;=\; (0.5/N)\,\mathrm{trace}\,(I) \;=\; 0.5.$$

We also consider the control effort determined by the H_2 norm from d to z_2 given by

$$\Pi_{\mathrm{ctr}} \;=\; (0.5/N)\,\mathrm{trace}\left(T^{-1}(rT^T T)\right) \;=\; 2r.$$

Thus, the control effort of spatially uniform feedback gain scales uniformly with the number of vehicles. Furthermore, the variance per vehicle for this type of actuation remains bounded when attention is paid to a microscopic performance measure (i.e., relative position errors). On the other hand, the variance per vehicle scales affinely with N when attention is paid to a macroscopic performance measure (i.e., absolute position errors). This phenomenon can lead to 'meandering' of the entire formation which can have undesirable consequences on throughput in automated highways [7].

B. Performance of optimal mistuning controllers

In this section, we examine the influence of platoon size on the performance of optimal mistuning controllers obtained in Section IV with $R = I$. This problem amounts to determining the scaling with N of macroscopic and microscopic performance measures Π_g and Π_l (cf. (Π)); and the control

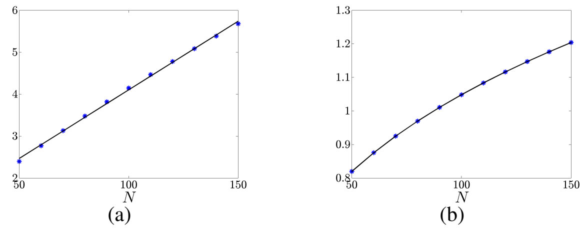


Fig. 3. (a) Affine scaling of $\Pi_g(F_l)$ (stars), $0.0327N + 0.8315$ (solid line); (b) quadratic logarithmic scaling of $\Pi_g(F_g)$ (stars), $0.0488(\log N)^2 - 0.0858 \log N + 0.4084$ (solid curve).

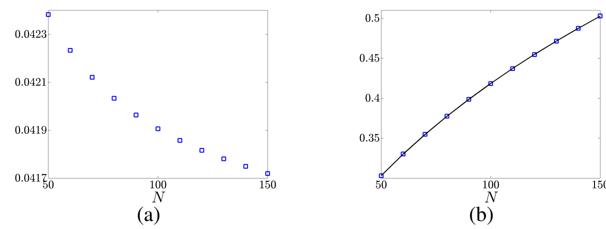


Fig. 5. (a) Scaling of control (squares) for F_l; (b) quadratic logarithmic scaling of control (squares) for F_g, $0.0376(\log N)^2 - 0.1540 \log N + 0.3297$ (solid curve).

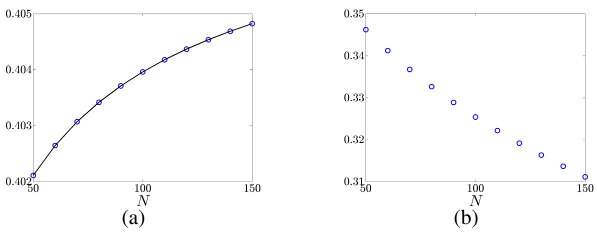


Fig. 4. (a) Quadratic logarithmic scaling of $\Pi_l(F_l)$ (circles), $-0.0005(\log N)^2 + 0.0069 \log N + 0.3827$ (solid curve); (b) scaling of $\Pi_l(F_g)$ (circles).

effort in platoons using controller F_g and F_l, respectively

$$\begin{aligned} \dot{x} &= (A - F_m C_2)\, x + d, \\ z &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} Q_s^{1/2} \\ -F_m C_2 \end{bmatrix} x, \qquad m, s = g, l. \end{aligned}$$

For example, $\Pi_g(F_l)$ denotes influence of optimal mistuning feedback gain obtained with $Q_l = T$ on microscopic performance measure, Q_g, and similarly for the other three cases.

Figure 3(a) shows affine scaling with N of the macroscopic performance measure, $\Pi_g(F_l)$, with optimal microscopic mistuning profile, F_l. On the other hand, $\Pi_g(F_g)$ appears to be scaling quadratically with $\log N$ (cf. Fig. 3(b)); this illustrates the capability of an optimally designed mistuning gain to dramatically improve the scaling of the macroscopic performance measure with the platoon size. Furthermore, the microscopic performance $\Pi_l(F_g)$ decreases relative to $\Pi_l(F_l)$ increasing with platoon size (see Fig. 4). However, the control effort for controller F_g increases quadratically with $\log N$ (cf. Fig. 5(b)) as opposed to control effort decreasing with platoon size for controller F_l (cf. Fig. 5(a)). This increase in control effort with N can be attributed to a gradual loss of controllability of strategies that use only relative position exchange in large-scale formations [5]. From a practical point of view, however, the optimal macroscopic mistuning controller F_g may represent a reasonable middle ground for achieving both satisfactory tightness and throughput requirements.

VI. Concluding remarks

We consider the design of H_2 optimal localized feedback gains for vehicular platoons. Motivated by the recent work of [8] on 'mistuning', we begin by considering stable platoons with nominal diffusion dynamics and searching for feedback gains that are small in norm. We do this by posing an expensive optimal control regime, and using perturbation analysis to decouple the necessary conditions for optimality. We derive analytical expressions for the structured gain for different objective functions, namely objectives that focus on the global performance of the platoon and those that focus on the local performance. We then remove the norm constraint from the feedback gain, and use the solution obtained from expensive control to initialize Newton's method.

One of the major conclusions of this work is that optimally designed spatially varying feedback gains can dramatically improve the scaling of macroscopic performance measures with platoon size. We have shown that our design converts an affine dependence on N of the macroscopic performance measure to a quadratic dependence on $\log N$. This may be paramount in applications involving large-scale vehicular formations.

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