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Brief paper

$\mathcal{H}_2$ norm of linear time-periodic systems: A perturbation analysis ☆

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Abstract

We consider a class of linear time-periodic systems in which the dynamical generator $A(t)$ represents the sum of a stable time-invariant operator A_0 and a small-amplitude zero-mean T-periodic operator $\epsilon A_p(t)$. We employ a perturbation analysis to develop a computationally efficient method for determination of the $\mathcal{H}_2$ norm. Up to second order in the perturbation parameter ϵ we show that: (a) the $\mathcal{H}_2$ norm can be obtained from a conveniently coupled system of Lyapunov and Sylvester equations that are of the same dimension as A_0; (b) there is no coupling between different harmonics of $A_p(t)$ in the expression for the $\mathcal{H}_2$ norm. These two properties do not hold for arbitrary values of ϵ, and their derivation would not be possible if we tried to determine the $\mathcal{H}_2$ norm directly without resorting to perturbation analysis. Our method is well suited for identification of the values of period T that lead to the largest increase/reduction of the $\mathcal{H}_2$ norm. Two examples are provided to motivate the developments and illustrate the procedure.

Keywords: Linear time-periodic systems; Distributed systems; Frequency responses; $\mathcal{H}_2$ norm; Perturbation analysis

1. Introduction

Time-periodic systems arise in many important physical and engineering problems (Nayfeh & Mook, 1979). Examples of finite-dimensional systems include the Hill and Mathieu equations, and examples of infinite-dimensional systems include the equations describing periodic excitations of fluids, beams, plates, strings, and membranes. *Floquet analysis* provides a theoretical framework for the investigation of local stability properties of these systems (Farkas, 1994). On the other hand, the so-called *lifting technique* (Bamieh & Pearson, 1992) and the *harmonic balance approach* (Wereley & Hall, 1990) are most suitable for the analysis of input–output properties of the linearized versions of these systems.

The utility of input–output analysis for linear time-invariant (LTI) systems is well documented (Zhou, Doyle, & Glover, 1996). The $\mathcal{H}_2$ norm is an appealing measure of input–output amplification, as it quantifies variance amplification in stochastically driven linear systems. For LTI systems, the $\mathcal{H}_2$ norm is determined by traces of controllability or observability Gramians which represent solutions to standard Lyapunov equations. On the other hand, the $\mathcal{H}_2$ norm of linear time-periodic (LTP) systems can be expressed in terms of the solution to the so-called *harmonic Lyapunov equation* (Zhou, Hagiwara, & Araki, 2003). Since the entries in this equation are bi-infinite matrices with, in general, operator-valued elements, the computation of the $\mathcal{H}_2$ norm of LTP systems is a nontrivial exercise. Furthermore, the state-transition matrix of most LTP systems is difficult to obtain (both analytically and numerically) which additionally hinders the analysis. The recent article (Zhou et al., 2003) addressed these problems by: (a) approximation of $A(t)$ in the state equation by piecewise constant functions; (b) truncation of bi-infinite matrices in the harmonic Lyapunov equation. However, for systems described by partial integro-differential equations (PIDEs) even this approach would require solving a large-scale Lyapunov equation; for an accurate computation of the $\mathcal{H}_2$ norm of PIDEs

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with one spatial variable, the entries into this equation are typically matrices with a large number of rows and columns.

In this paper, we study LTP systems in which $A(t)$ is given by the sum of a stable time-invariant operator A_0 and a small-amplitude zero-mean T-periodic operator $\epsilon A_p(t)$. For example, these systems can be obtained by linearization of time-invariant nonlinear systems around small-amplitude T-periodic trajectories. We employ a perturbation analysis to develop a computationally efficient method for determining the $\mathcal{H}_2$ norm. Up to second order in the perturbation parameter ϵ we show that: (a) the $\mathcal{H}_2$ norm can be obtained from a conveniently coupled system of *finite-dimensional* Lyapunov and Sylvester equations; (b) there is no coupling between different harmonics of $A_p(t)$ in the expression for the $\mathcal{H}_2$ norm. These two properties do not hold for arbitrary values of ϵ, and their derivation would not be possible if we tried to determine the $\mathcal{H}_2$ norm directly without resorting to perturbation analysis. We would like to emphasize that the dimension of the Lyapunov and Sylvester equations mentioned above is *the same as* the dimension of $A(t)$. This is in the same spirit as the perturbation analysis performed in Ndiaye and Sorine (2000). Such finite-dimensional equations are clearly easier to solve than the harmonic Lyapunov equation, which is composed of bi-infinite matrices each element of which has the same dimension as $A(t)$. Our perturbation method is well suited for identification of the values of period T that lead to the largest increase/reduction of the $\mathcal{H}_2$ norm. An immediate application domain is in fluid mechanics where temporally-periodic excitations can be introduced either to suppress turbulence (Jovanović, 2006, 2008) or to enhance mixing.

We note that the perturbation analysis used here has strong parallels with the approach of Fardad and Bamieh (2008, 2005) for the $\mathcal{H}_2$ analysis of linear *spatially-periodic* systems. However, there are some important differences in the structure of frequency response operators for temporally- and spatially-periodic systems which necessitates separate treatments. For example, in spatially-periodic systems one often encounters cascades of spatially-invariant differential and spatially-periodic multiplication operators which somewhat complicate their analysis (Fardad, 2006; Fardad, Jovanović, & Bamieh, in press). On the other hand, state-space models of LTP systems do not contain cascades of differential and periodic operators, which imposes some additional structure that can be utilized to simplify the analysis.

Our presentation is organized as follows: In Section 2 we formulate the problem and provide two examples that serve as a motivation for our analysis. In Section 3 we give a brief overview of the notion of the *frequency response* for exponentially stable LTP systems. In Section 4 we define the $\mathcal{H}_2$ norm for LTP systems. In Section 5 we employ perturbation analysis to develop an efficient procedure for computing the $\mathcal{H}_2$ norm of systems subject to small-amplitude oscillations. In Section 6, we use the developed method to determine the second-order corrections to the $\mathcal{H}_2$ norms of systems described in Section 2.1 and Section 2.2. In Section 7, we end our presentation with some concluding remarks.

2. Problem formulation and motivating examples

Let a linear dynamical system be given by its state-space representation

$$\partial_t \psi = A(t)\psi + Bd, \tag{1a}$$

$$\phi = C\psi, \tag{1b}$$

where ψ, ϕ, and d, respectively, denote the state, output, and input vector-valued functions. Eqs. (1a) and (1b) are also commonly referred to as an *evolution system* (Pazy, 1983). We assume that $A(t)$ represents a time-periodic operator with period $T = 2\pi/\omega_o$, $A(t) = A(t+T)$, that generates an exponentially stable strongly-continuous (C_0) semigroup on a Hilbert space $\mathbb{H}$. The input and output operators B and C are assumed to be time-invariant. In the case where A, B, and C are unbounded operators we assume that ψ and d belong to dense subsets of appropriate Hilbert spaces.

In this paper, we consider a class of LTP systems in which the operator $A(t)$ can be represented as

$$A(t) = A_0 + \epsilon A_p(t),$$

where ϵ is a small parameter, A_0 is a time-invariant operator and creates an exponentially stable evolution, and $A_p(t)$ is a zero-mean T-periodic operator. In other words, we assume that $A_p(t)$ can be expanded in its Fourier series, $A_p(t) = \sum_{n\in\mathbb{Z}\setminus 0} A_n e^{jn\omega_o t}$. Our objective is to derive a computationally efficient procedure for the determination of the $\mathcal{H}_2$ norm of system (1) using a perturbation analysis.

We are particularly interested in distributed systems with one spatial variable $y \in [-1, 1]$. To highlight this, we rewrite (1) as

$$\partial_t \psi(y,t) = A(t)\psi(y,t) + Bd(y,t), \tag{2a}$$

$$\phi(y,t) = C\psi(y,t), \tag{2b}$$

where for each t, $\psi(\cdot,t)$, $\phi(\cdot,t)$, and $d(\cdot,t)$ denote vector-valued fields in $L^2[-1, 1]$. We assume that $\psi(\cdot,t)$ belongs to some dense subset $D(A)$ of sufficiently smooth functions in $L^2[-1, 1]$ for every t. On the other hand, $A(t)$, B, and C are linear (integro-differential) operators in y, with $A(t) = A(t+T)$. The example presented in Section 2.2 illustrates the structure of these operators for a system describing the evolution of velocity perturbations in a two-dimensional oscillating channel flow. We note that with a careful choice of notation and precise definition of underlying signals, operators, and spaces, all of our results hold for both finite-dimensional LTP systems and infinite-dimensional LTP systems described by (2).

Let H denote the mapping from input d to output ϕ, $\phi = Hd$. We assume that H has a kernel representation given by

$$\phi(t) = \int_0^t H(t,\tau)d(\tau)\mathrm{d}\tau,$$

for finite-dimensional systems of the form (1), and

$$\phi(y,t) = \int_0^t \int_{-1}^1 H(y,\eta;t,\tau)d(\eta,\tau)\mathrm{d}\eta\mathrm{d}\tau,$$

for infinite-dimensional systems of the form (2). Here, with an abuse of notation, we use the same symbol for an operator and its kernel function. It is not difficult to show that the kernel function representing H is a doubly-periodic function in t and τ, i.e. $H(y, \eta; t, \tau) = H(y, \eta; t+nT, \tau+nT)$, $n \in \mathbb{N}_0$ (Bamieh & Pearson, 1992). We assume that the system is L^2-stable and that its $\mathcal{H}_2$ norm is well defined; detailed conditions are given in assumptions (A1)–(A3) in Section 4.

We next provide two examples that serve as a motivation for our analysis. The first example represents a dissipative version of the well-known Mathieu equation, and the second example describes the dynamics of flow fluctuations in a two-dimensional channel flow subject to a streamwise pressure gradient and an oscillatory motion of the lower wall.

2.1. The dissipative Mathieu equation

The forced dissipative Mathieu equation is given by

$$\ddot{x} + 2b\dot{x} + (a - 2\epsilon \cos \omega_o t)x = d,$$

where a and b denote positive parameters. By selecting

$$\psi(t) := [\, x(t) \quad \dot{x}(t) \,]^{\mathrm{T}}, \qquad \phi(t) := x(t),$$

this equation can be represented by (1) with

$$A(t) := \begin{bmatrix} 0 & 1 \\ -(a - 2\epsilon \cos \omega_o t) & -2b \end{bmatrix},$$

$$B := [0 \quad 1]^{\mathrm{T}}, \qquad C := [1 \quad 0].$$

Clearly, in this example $\mathbb{H} := \mathbb{R}^2$, and

$$\begin{aligned} A(t) &:= A_0 + \epsilon A_p(t) \\ &= A_0 + \epsilon \left(A_{-1} \mathrm{e}^{-\mathrm{j}\omega_o t} + A_1 \mathrm{e}^{\mathrm{j}\omega_o t} \right), \end{aligned}$$

where

$$A_0 := \begin{bmatrix} 0 & 1 \\ -a & -2b \end{bmatrix}, \qquad A_{\pm 1} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

In Section 6.1 we consider small-amplitude oscillations and use the perturbation analysis of Section 4 to determine how the $\mathcal{H}_2$ norm changes with forcing frequency ω_o.

2.2. An example from fluid mechanics

We consider the dynamics of velocity fluctuations in a two-dimensional channel flow with geometry illustrated in Fig. 1. For background material on the use of input–output norms in analysis of fluid systems see Schmid and Henningson (2001), Bamieh and Dahleh (2001) and Jovanović and Bamieh (2005) and the references therein.

Incompressible flow of a viscous Newtonian fluid satisfies the Navier-Stokes (NS) and the continuity equations given in their non-dimensional forms by Panton (1996)

$$\mathbf{u}_t = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla P + (1/R)\Delta\mathbf{u} + \mathbf{F},$$
$$0 = \nabla\cdot\mathbf{u},$$

where $\mathbf{u}$ is the velocity vector, P is the pressure, $\mathbf{F}$ is the body force, ∇ is the gradient, and $\Delta := \nabla^2$ is the two-dimensional Laplacian. The Reynolds number R is defined in terms of the centerline velocity and the channel half-width.

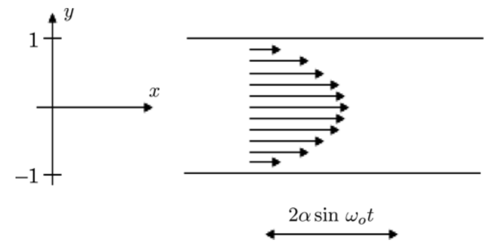

Fig. 1. A two-dimensional channel flow subject to a streamwise pressure gradient and an oscillatory motion of the lower wall.

Let the flow be subject to a streamwise pressure gradient, $P_x = -2/R$, an oscillatory motion of the lower wall, $U(y = -1, t) = 2\alpha \sin \omega_o t$, and let the nominal body force be equal to zero, $\bar{\mathbf{F}} \equiv 0$. Here, t denotes the non-dimensional time, α and ω_o are, respectively, the non-dimensional amplitude and frequency of the wall oscillations, and U is the nominal streamwise velocity. In steady-state the NS equations simplify to the x-direction momentum equation

$$U_t = -P_x + (1/R)U_{yy},$$

subject to

$$U(-1, t) = 2\alpha \sin \omega_o t, \qquad U(1) = 0, \qquad P_x = -2/R.$$

With the appropriate scaling of the NS equations, α and ω_o can be expressed as $\{\alpha = R_u/R, \omega_o = \Omega/R\}$, where R_u is the Reynolds number defined in terms of the wall oscillation amplitude (in physical units) and the channel half-width, and Ω is the Stokes number (i.e., the non-dimensional temporal frequency). Under these conditions, it is readily shown that the steady-state solution of the NS equations is given by

$$U(y, t) = U_0(y) + 2(R_u/R)U_1(y, t), \qquad U_0(y) = 1 - y^2,$$
$$U_1(y, t) = U_c(y) \cos(\Omega/R)t + U_s(y) \sin(\Omega/R)t,$$

where $U_c(y)$ and $U_s(y)$ represent solutions to

$$U_s''(y) = -\Omega U_c(y), \qquad U_c''(y) = \Omega U_s(y),$$
$$U_c(\pm 1) = U_s(1) = 0, \qquad U_s(-1) = 1.$$

Here, $U_r''(y)$ denotes the second derivative of $U_r(y)$, that is $U_r''(y) := \mathrm{d}^2 U_r(y)/\mathrm{d}y^2$, $r = s$ or $r = c$.

The linearization of the NS equations around $U(y, t)$ in combination with the Fourier transform in x yields a one-dimensional PIDE (in y) parameterized by the wave-number $k_x \in \mathbb{R}$. This PIDE has a state-space representation (2), with the state of the system determined by a scalar field $\psi(k_x, y, t)$ denoting the stream function (Panton, 1996). On the other hand, the input and output fields d and ϕ, respectively, denote body force and velocity fluctuations. With this choice of the system's state, the operator $A(t)$ in (2) is given by

$$A(t) := \Delta^{-1} \left(\frac{1}{R} \Delta^2 + \mathrm{j}k_x \left(U''(y, t) - U(y, t)\Delta \right) \right),$$

where $\Delta := \partial_{yy} - k_x^2$ with homogeneous Dirichlet boundary conditions, and $\Delta^2 := \partial_{yyyy} - 2k_x^2 \partial_{yy} + k_x^4$ with homogeneous

Dirichlet *and* Neumann boundary conditions. Note that Δ^{-1} is defined by

$$\Delta^{-1} : f \mapsto g \Leftrightarrow f = \Delta g =: g'' - k_x^2 g, \qquad g(\pm 1) = 0.$$

The underlying Hilbert space for A is given by Reddy and Henningson (1993)

$$\mathbb{H} := \left\{ g \in L^2[-1,1];\, g'' \in L^2[-1,1],\, g(\pm 1) = 0 \right\}.$$

The operator A is unbounded and is defined on the domain

$$D(A) := \left\{ g \in \mathbb{H};\, g^{(4)} \in L^2[-1,1],\, g'(\pm 1) = 0 \right\}.$$

If $\mathbb{H}$ is endowed with the inner product that determines the kinetic energy of velocity fluctuations, then $BB^* = C^*C = I$, and the adjoint of the operator $A(t)$ is given by Jovanović and Bamieh (2005)

$$A^*(t) = (1/R)\Delta^{-1}\Delta^2 + \mathrm{j}k_x(U(y,t) - \Delta^{-1}U''(y,t)).$$

Finally, we represent $A(t)$ in a form suitable for $\mathcal{H}_2$ norm analysis

$$A(t) = A_0 + (R_u/R)(A_{-1}\mathrm{e}^{-\mathrm{j}(\Omega/R)t} + A_1\mathrm{e}^{\mathrm{j}(\Omega/R)t}),$$

where

$$A_0 := \Delta^{-1}\left(\frac{1}{R}\Delta^2 + \mathrm{j}k_x\left(U_0''(y) - U_0(y)\Delta\right)\right),$$
$$A_{\pm 1} := A_c \mp \mathrm{j}A_s,$$
$$A_r := \mathrm{j}k_x\Delta^{-1}\left(U_r''(y) - U_r(y)\Delta\right), \quad r = s, c.$$

In Section 6.2, we consider wall oscillations of small amplitude ($R_u \ll R$), and determine the $\mathcal{H}_2$ norm dependence on k_x and Ω in channel flow with $R = 2000$ using the perturbation analysis of Section 4.

3. Frequency response of LTP systems

We next provide a brief overview of the notion of the *frequency response* for exponentially stable LTP systems with period $T = 2\pi/\omega_o$. We refer the reader to Sandberg, Möllerstedt, and Bernhardsson (2005), Wereley and Hall (1990), Zhou and Hagiwara (2002) and Zhou et al. (2003) for additional information. In particular, the details of rigorous conditions for the existence of frequency response operators of the LTP systems can be found in Zhou and Hagiwara (2002).

It is a standard fact that the frequency response of a stable LTI system describes how a persistent harmonic input of a certain frequency propagates through the system in steady state. In other words, the steady-state response of a stable LTI system to an input signal of frequency ω, is a periodic signal of the same frequency but with a modified amplitude and phase. The amplitude and phase of the output signal are precisely determined by the value of the frequency response at the input frequency ω.

On the other hand, the steady-state response of a stable LTP system to a harmonic input of frequency ω contains an infinite number of harmonics separated by integer multiples of ω_o, that is $\omega + n\omega_o$, $n \in \mathbb{Z}$. Using this fact and the analogy with the LTI systems, the frequency response of an LTP system can be defined by introducing the notion of *exponentially modulated periodic* (EMP) signals. As shown in Wereley and Hall (1990), EMP signals are more suitable for the analysis of LTP systems than persistent complex exponentials. Namely, the steady-state response of (1) to an EMP signal

$$d(t) = \sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\mathrm{j}(n\omega_o + \theta)t}, \quad \theta \in [0, \omega_o),$$

is also an EMP signal

$$\phi(t) = \sum_{n=-\infty}^{\infty} \phi_n \mathrm{e}^{\mathrm{j}(n\omega_o + \theta)t}, \quad \theta \in [0, \omega_o).$$

The frequency response of (1) is an operator that maps a bi-infinite vector $\mathbf{d} := \mathrm{col}\,\{d_n\}_{n\in\mathbb{Z}}$ to a bi-infinite vector $\boldsymbol{\phi} := \mathrm{col}\,\{\phi_n\}_{n\in\mathbb{Z}}$, that is $\boldsymbol{\phi} = \mathcal{H}_\theta \mathbf{d}$.

For system (1), the frequency response operator $\mathcal{H}_\theta$ can be expressed as (Wereley, 1991)

$$\mathcal{H}_\theta = \mathcal{C}(\mathcal{E}(\theta) - \mathcal{A})^{-1}\mathcal{B},$$

where $\mathcal{E}(\theta)$ is a block-diagonal operator given by $\mathcal{E}(\theta) := \mathrm{diag}\{\mathrm{j}(\theta + n\omega_o)I\}_{n\in\mathbb{Z}}$, and I is the identity operator. On the other hand, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ represent block-Toeplitz operators, e.g.

$$\mathcal{A} := \mathrm{toep}\{\ldots, A_2, A_1, \boxed{A_0}, A_{-1}, A_{-2}, \ldots\},$$

where the box denotes the element on the main diagonal of $\mathcal{A}$. This bi-infinite matrix representation is obtained by expanding the operators A, B, and C of (1) in their Fourier series, e.g. $A(t) = \sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{\mathrm{j}n\omega_o t}$. Clearly, since B and C are time-invariant operators their block-Toeplitz representations simplify to block-diagonal representations, i.e., $\mathcal{B} = \mathrm{diag}\{B\}$ and $\mathcal{C} = \mathrm{diag}\{C\}$.

4. $\mathcal{H}_2$ norm of LTP systems

The $\mathcal{H}_2$ norm of a T-periodic system (1) with $\phi = Hd$ is defined as (Bamieh & Pearson, 1992)

$$\|\mathcal{H}\|_2^2 := \frac{1}{T}\int_0^T \int_0^\infty \left[\|H\|_{HS}^2\right](t,\tau)\,\mathrm{d}t\,\mathrm{d}\tau,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt (HS) norm. For finite-dimensional systems with kernel function $H(t,\tau)$, the HS norm simplifies to the Frobenius norm of matrices

$$\left[\|H\|_{HS}^2\right](t,\tau) := \mathrm{tr}\left(H^*(t,\tau)H(t,\tau)\right), \tag{3}$$

where $\mathrm{tr}(\cdot)$ denotes the matrix trace. For infinite-dimensional systems with spatio-temporal kernel function $H(y,\eta;t,\tau)$, the HS norm is given by

$$\left[\|H\|_{HS}^2\right](t,\tau) := \int_{-1}^1 \int_{-1}^1 \mathrm{tr}\left(H^*(y,\eta;t,\tau)H(y,\eta;t,\tau)\right)\mathrm{d}\eta\,\mathrm{d}y.$$

As shown in Bamieh and Pearson (1992), the $\mathcal{H}_2$ norm of an LTP system can be interpreted as the square-average of the L^2

norms of the responses to a set of impulse forcing functions applied over the entire interval $[0, T]$. This interpretation of the $\mathcal{H}_2$ norm of LTP systems represents the appropriate generalization of a well-known deterministic interpretation of the $\mathcal{H}_2$ norm of LTI systems (Zhou et al., 1996). The stochastic interpretation of the $\mathcal{H}_2$ norm is discussed in the Appendix.

Notation: We use trace$(\cdot)$ to denote the trace of an infinite-dimensional operator with spatial kernel function G

$$\text{trace}(G) := \int_{-1}^{1} \text{tr}(G(y, y))\mathrm{d}y.$$

In order to unify our notation between the trace of operators and the trace of matrices, we assume that trace(G) collapses to the standard matrix trace tr(G) when G is finite dimensional. It can be shown that

$$\int_{-1}^{1}\int_{-1}^{1} \text{tr}\left(G^*(y,\eta)G(y,\eta)\right)\mathrm{d}\eta\mathrm{d}y = \int_{-1}^{1}\text{tr}(\widetilde{G}(y,y))\mathrm{d}y = \text{trace}(\widetilde{G}),$$

where $\widetilde{G} = G^*G$ is the *kernel composition* of G^* and G.

It is now possible to give a unified definition of the HS norm for both finite- and infinite-dimensional systems by replacing (3) with

$$\left[\|H\|_{HS}^2\right](t,\tau) := \text{trace}(H^*(t,\tau)H(t,\tau)), \tag{4}$$

where for infinite-dimensional systems $H(t, \tau)$ is a spatial kernel function at every (t, τ).

Let $H(t, \tau) = C\,\Phi(t, \tau)B$, where Φ is the state-transition operator. We state the main assumptions on systems (1)–(2) as follows:

(A1) $A(t)$ describes an exponentially stable evolution;
(A2) BB^* and C^*C are bounded operators;
(A3) $\int_0^\infty \left[\|\Phi\|_{HS}^2\right](t,\tau)\mathrm{d}t$ is finite for every $\tau \in [0, T]$.

Conditions (A1)–(A3) guarantee that the $\mathcal{H}_2$ norm of the LTP system is well-defined and finite.

The $\mathcal{H}_2$ norm of an LTP system can also be found from (Colaneri, 2000)

$$\|\mathcal{H}\|_2^2 = \frac{1}{T}\int_0^T \text{trace}(V(\tau)C^*C)\,\mathrm{d}\tau \tag{5}$$

$$= \frac{1}{T}\int_0^T \text{trace}(W(\tau)BB^*)\,\mathrm{d}\tau, \tag{6}$$

where $V(\cdot)$, $W(\cdot)$ are the T-periodic steady-state solutions of the following *differential Lyapunov equations* (DLEs)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V(\tau) = V(\tau)A^*(\tau) + A(\tau)V(\tau) + BB^*, \tag{7}$$

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}W(\tau) = A^*(\tau)W(\tau) + W(\tau)A(\tau) + C^*C. \tag{8}$$

Let $\{W_n\}_{n\in\mathbb{Z}}$ denote the Fourier series coefficients of W,

$$W(t) = \sum_{n\in\mathbb{Z}} W_n\,\mathrm{e}^{\mathrm{j}n\omega_o t}.$$

Eq. (6) demonstrates that the $\mathcal{H}_2$ norm is equal to the trace of the *constant* component W_0BB^* of the T-periodic function $W(\cdot)BB^*$. Let $\mathcal{W}\mathcal{B}\mathcal{B}^*$ denote the Toeplitz representation of $W(\cdot)BB^*$. Then W_0BB^* constitutes the diagonal element of $\mathcal{W}\mathcal{B}\mathcal{B}^*$. This motivates yet another method for the computation of the $\mathcal{H}_2$ norm in which (8) is replaced by its Toeplitz counterpart, an equation in $\mathcal{W}$, referred to as the *harmonic Lyapunov equation* (HLE) (Zhou et al., 2003). We summarize this method in the next theorem.

Theorem 1. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ denote the Toeplitz representations of $A(t)$, B, and C, respectively, with*

$$\mathcal{F} := \mathcal{A} - \mathcal{E}(0) = \mathcal{A} - \text{diag}\{\mathrm{j}n\omega_o I\}_{n\in\mathbb{Z}}.$$

Then under assumptions (A1)–(A3)

$$\|\mathcal{H}\|_2^2 = \text{trace}(V_0C^*C) = \text{trace}(W_0BB^*),$$

where V_0 and W_0 are, respectively, the diagonal elements of the Toeplitz operators $\mathcal{V}$ and $\mathcal{W}$ that satisfy the harmonic Lyapunov equations

$$\mathcal{F}\mathcal{V} + \mathcal{V}\mathcal{F}^* = -\mathcal{B}\mathcal{B}^*, \tag{9a}$$

$$\mathcal{F}^*\mathcal{W} + \mathcal{W}\mathcal{F} = -\mathcal{C}^*\mathcal{C}. \tag{9b}$$

For LTI systems, the above formulae simplify to the well-known expressions commonly used for determination of the $\mathcal{H}_2$ norm (Zhou et al., 1996).

5. Perturbation analysis of the $\mathcal{H}_2$ norm

Since the entries of the harmonic Lyapunov equation are bi-infinite matrices with, in general, operator-valued elements, determination of the $\mathcal{H}_2$ norm of the LTP systems is arguably a computationally intensive undertaking. In view of this, we will consider the problem where the operator $A(t)$ can be represented as the sum of a time-invariant operator A_0 and a zero-mean time-periodic operator $\epsilon A_p(t)$, where ϵ denotes a small real parameter. For this special case, we will employ a perturbation analysis to develop a computationally efficient method for the determination of the $\mathcal{H}_2$ norm. We will show that the $\mathcal{H}_2$ norm can be obtained by solving a conveniently coupled system of Lyapunov and Sylvester equations. The entries in these equations are determined by the elements of the bi-infinite matrices in (9).[1]

Using the structure of $A(t)$, we represent the operator $\mathcal{F}$ in (9) as

$$\mathcal{F} = \mathcal{F}_0 + \epsilon\sum_{m\in\mathbb{N}}\mathcal{F}_m,$$

$$\mathcal{F}_0 := \text{diag}\{F(n)\}_{n\in\mathbb{Z}} = \text{diag}\{A_0 - \mathrm{j}n\omega_o I\}_{n\in\mathbb{Z}}.$$

[1] For the oscillating channel flow example, the underlying Lyapunov and Sylvester equations are operator-valued equations in the wall-normal direction (y). A discretization in y can be used to obtain a set of matrix-valued equations that can be easily solved in e.g. MATLAB. The order of these equations is determined by the size of discretization in the wall-normal direction (typically around 50).

On the other hand, for each $m \in \mathbb{N}$, $\mathcal{F}_m$ represents a block-Toeplitz operator with A_{-m} and A_m on the mth upper and lower block sub-diagonals, respectively. For example,

$$\mathcal{F}_1 := \text{toep}\left\{\ldots, 0, A_1, \boxed{0}, A_{-1}, 0, \ldots\right\},$$

$$\mathcal{F}_2 := \text{toep}\left\{\ldots, 0, A_2, 0, \boxed{0}, 0, A_{-2}, 0, \ldots\right\},$$

and similarly for the other $\mathcal{F}_m$'s. In view of the above decomposition of $\mathcal{F}$, we rewrite the harmonic Lyapunov equation (9a)

$$\left(\mathcal{F}_0 + \epsilon \sum_{m\in\mathbb{N}} \mathcal{F}_m\right)\mathcal{V} + \mathcal{V}\left(\mathcal{F}_0^* + \epsilon \sum_{m\in\mathbb{N}} \mathcal{F}_m^*\right) = -\mathcal{B}\mathcal{B}^*,$$

and represent $\mathcal{V}$ as

$$\mathcal{V} := \sum_{n\in\mathbb{N}_0} \epsilon^n \mathcal{V}_n = \mathcal{V}_0 + \epsilon\mathcal{V}_1 + \epsilon^2\mathcal{V}_2 + \cdots. \tag{10}$$

The self-adjoint block-Toeplitz operators $\{\mathcal{V}_n\}_{n\in\mathbb{N}_0}$ satisfy the following sequence of operator Lyapunov equations

$$\mathcal{F}_0\mathcal{V}_0 + \mathcal{V}_0\mathcal{F}_0^* = -\mathcal{B}\mathcal{B}^*, \tag{11a}$$

$$\mathcal{F}_0\mathcal{V}_i + \mathcal{V}_i\mathcal{F}_0^* = -\sum_{m\in\mathbb{N}} (\mathcal{F}_m\mathcal{V}_{i-1} + \mathcal{V}_{i-1}\mathcal{F}_m^*), \tag{11b}$$

for each $i \in \mathbb{N}$. The unperturbed system is stable and thus from the necessity part of Theorem 2 of Zhou, Hagiwara, and Araki (2002) the operators $\mathcal{V}_0$ and $\mathcal{V}_i$, $i \in \mathbb{N}$ will always exist. Since $\mathcal{F}_0$ and $\mathcal{B}$ are block-diagonal operators, it follows from (11a) that $\mathcal{V}_0$ is also a block-diagonal operator, $\mathcal{V}_0 = \text{diag}\{X\}$, with

$$A_0X + XA_0^* = -BB^*.$$

Using linearity of (11b) we express $\mathcal{V}_1$ as

$$\mathcal{V}_1 = \sum_{m\in\mathbb{N}} \mathcal{V}_1^{(m)},$$

where

$$\mathcal{F}_0\mathcal{V}_1^{(m)} + \mathcal{V}_1^{(m)}\mathcal{F}_0^* = -(\mathcal{F}_m\mathcal{V}_0 + \mathcal{V}_0\mathcal{F}_m^*). \tag{12}$$

Here, for each $m \in \mathbb{N}$, $\mathcal{V}_1^{(m)}$ denotes a self-adjoint block-Toeplitz operator with non-zero elements on the mth block sub-diagonals; this structure of $\mathcal{V}_1^{(m)}$ follows directly from (12) and the simple observation that a product between a block-diagonal and the block-Toeplitz operator with non-zero elements on the mth block sub-diagonals yields an operator with non-zero elements on the mth block sub-diagonals. Thus, $\mathcal{V}_1$ is a trace-less operator, and each $\mathcal{V}_1^{(m)}$ is a self-adjoint block-Toeplitz operator with Y_m on the mth upper block sub-diagonal. Furthermore, the operator Y_m represents the solution to the following Sylvester equation

$$(A_0 + \mathrm{j}m\omega_o I)Y_m + Y_mA_0^* = -\left(A_{-m}X + XA_m^*\right).$$

Based on the linearity of (11b) and the above representation of $\mathcal{V}_1$ it follows that $\mathcal{V}_2$ can be expressed as

$$\mathcal{V}_2 = \sum_{m\in\mathbb{N}}\sum_{k\in\mathbb{N}} \mathcal{V}_2^{(m,k)},$$

where

$$\mathcal{F}_0\mathcal{V}_2^{(m,k)} + \mathcal{V}_2^{(m,k)}\mathcal{F}_0^* = -(\mathcal{F}_m\mathcal{V}_1^{(k)} + \mathcal{V}_1^{(k)}\mathcal{F}_m^*). \tag{13}$$

Now, since $\mathcal{F}_m$ and $\mathcal{V}_1^{(k)}$ are, respectively, block-Toeplitz operators with non-zero elements on the mth and kth block sub-diagonals, their product will have non-zero elements on the main-diagonal if and only if $m = k$. Thus, we see from (13) that only operators $\mathcal{V}_2^{(m,m)}$ have a non-zero trace; for $m \neq k$, the operators $\mathcal{V}_2^{(m,k)}$ are trace-less. In view of this, we disjoin the block-diagonal part of $\mathcal{V}_2^{(m,m)}$ from the rest of it

$$\mathcal{V}_2^{(m,m)} = \text{diag}\{Z_m\} + \bar{\mathcal{V}}_2^{(m,m)},$$

and derive the following Lyapunov equation for operator Z_m

$$A_0Z_m + Z_mA_0^* = -(A_mY_m + Y_m^*A_m^* + A_{-m}Y_m^* + Y_mA_{-m}^*).$$

From (10), the fact that $\mathcal{V}_1$ is trace-less, and the form of $\mathcal{V}_2$ shown above, we conclude that up to second order in the perturbation parameter ϵ the $\mathcal{H}_2$ norm can be expressed as

$$\|\mathcal{H}\|_2^2 = \text{trace}\left(\left(X + \epsilon^2 \sum_{m\in\mathbb{N}} Z_m\right)C^*C\right) + \text{higher-order terms in } \epsilon. \tag{14}$$

Remark 1. Expression (14) is found for a system with an infinite number of harmonics in its A operator, $A_p(t) = \sum_{n\in\mathbb{Z}\backslash 0} A_n \mathrm{e}^{\mathrm{j}n\omega_o t}$. We emphasize that (14) represents a formal expansion of the $\mathcal{H}_2$ norm and we make no claims with regard to its convergence. In the case of a *finite* number of harmonics in operator A, $A_p(t) = \sum_{n\in\mathbb{M}} A_n\mathrm{e}^{\mathrm{j}n\omega_o t}$, $\mathbb{M} := \{-M, -M+1, \ldots, M-1, M\}\backslash 0$, it can be shown that the convergence of the $\mathcal{H}_2$ norm perturbation series is guaranteed by the exponential stability of the unperturbed system and the boundedness of the operators BB^* and C^*C. We omit the details for sake of brevity and refer the reader to Fardad (2006, Appendix to Chap. 5) for a similar proof.

Based on the preceding discussion we state the following result.

Theorem 2. *For a finite number of harmonics, up to second order in the perturbation parameter ϵ, the $\mathcal{H}_2$ norm of system (1) with*

$$A(t) = A_0 + \epsilon \sum_{n\in\mathbb{M}} A_n\mathrm{e}^{\mathrm{j}n\omega_o t},$$

$$\mathbb{M} := \{-M, -M+1, \ldots, M-1, M\} \setminus 0,$$

is given by

$$\|\mathcal{H}\|_2^2 = \text{trace}\left(\left(X + \epsilon^2 \sum_{m=1}^{M} Z_m\right)C^*C\right) + O(\epsilon^3),$$

where

$$A_0X + XA_0^* = -BB^*,$$

$$(A_0 + \mathrm{j}m\omega_o I)Y_m + Y_mA_0^* = -\left(A_{-m}X + XA_m^*\right),$$

$$A_0Z_m + Z_mA_0^* = -(A_mY_m + Y_m^*A_m^* + A_{-m}Y_m^* + Y_mA_{-m}^*).$$

Remark 2. Up to second order in the perturbation parameter ϵ, there is no coupling between different harmonics of $A_p(t)$ in the expression for the $\mathcal{H}_2$ norm. This decoupling between different harmonics does not hold for arbitrary values of ϵ, and its derivation would not be possible if we tried to solve the harmonic Lyapunov equation directly without resorting to perturbation analysis.

When $A(t)$ contains only the first harmonic ω_o, i.e.,

$$A(t) = A_0 + \epsilon\left(A_{-1}e^{-j\omega_o t} + A_1 e^{j\omega_o t}\right),$$

the operator $\mathcal{F}$ can be represented as $\mathcal{F} = \mathcal{F}_0 + \epsilon\mathcal{F}_1$, where $\mathcal{F}_0$ and $\mathcal{F}_1$ are defined in the beginning of this section. Using the structure of operators $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{V}_{i-1}$ in (11b) we can establish that:

- For any $n \in \mathbb{N}_0$, $\mathcal{V}_{2n}$ in (10) is a self-adjoint block-Toeplitz operator with non-zero elements on block sub-diagonals $2k$, $k = 0, \ldots, n$, that is

$$\mathcal{V}_{2n} = \operatorname{diag}\{V_{2n,0}\} + \sum_{k=1}^{n} \mathcal{S}_{2k}\operatorname{diag}\{V_{2n,2k}\} + \sum_{k=1}^{n}\operatorname{diag}\{V_{2n,2k}^*\}\,\mathcal{S}_{2k}^*,$$

  where $\mathcal{S}_{2k}$ denotes a bi-infinite block-Toeplitz operator with identity operators on the upper block sub-diagonal $2k$. Notation $V_{n,k}$ indicates that $V_{n,k}$ belongs to the kth upper block sub-diagonal of $\mathcal{V}_n$, and, for any $\{n \in \mathbb{N}_0, k = 0, \ldots, n\}$, the operators $V_{2n,2k}$ represent the solutions to Lyapunov and Sylvester equations given in Theorem 3.
- For any $n \in \mathbb{N}$, $\mathcal{V}_{2n-1}$ in (10) is a self-adjoint block-Toeplitz operator with non-zero elements on block sub-diagonals $2k-1$, $k = 1, \ldots, n$, that is

$$\mathcal{V}_{2n-1} = \sum_{k=1}^{n} \mathcal{S}_{2k-1}\operatorname{diag}\{V_{2n-1,2k-1}\} + \sum_{k=1}^{n}\operatorname{diag}\{V_{2n-1,2k-1}^*\}\,\mathcal{S}_{2k-1}^*,$$

  where $\mathcal{S}_{2k-1}$ denotes a bi-infinite block-Toeplitz operator with identity operators on the upper block sub-diagonal $2k-1$. Notation $V_{n,k}$ indicates that $V_{n,k}$ belongs to the kth upper block sub-diagonal of $\mathcal{V}_n$. Thus, $\operatorname{trace}(\mathcal{V}_{2n-1}) \equiv 0$, and, for any $\{n \in \mathbb{N}, k = 1, \ldots, n\}$, the operators $V_{2n-1,2k-1}$ represent the solutions to Lyapunov and Sylvester equations given in Theorem 3.

The above observations for time-periodic operators $A(t)$ with a single harmonic ω_o are summarized in Theorem 3.

Theorem 3. *The $\mathcal{H}_2$ norm of system (1) with*

$$A(t) = A_0 + \epsilon\left(A_{-1}e^{-j\omega_o t} + A_1 e^{j\omega_o t}\right), \quad 0 < \epsilon \ll 1,$$

is given by

$$\|\mathcal{H}\|_2^2 = \sum_{n=0}^{\infty} \epsilon^{2n}\operatorname{trace}\left(V_{2n,0}C^*C\right),$$

where

$$A_0V_{0,0} + V_{0,0}A_0^* = -BB^*,$$

$$A_0V_{2n,0} + V_{2n,0}A_0^* = -(A_1V_{2n-1,1} + V_{2n-1,1}^*A_1^* + A_{-1}V_{2n-1,1}^* + V_{2n-1,1}A_{-1}^*),$$

$$(A_0 + jl\omega_o I)V_{l,l} + V_{l,l}A_0^* = -(A_{-1}V_{l-1,l-1} + V_{l-1,l-1}A_1^*), \quad l \in \mathbb{N},$$

$$(A_0 + jm\omega_o I)V_{l,m} + V_{l,m}A_0^* = -(A_{-1}V_{l-1,m-1} + V_{l-1,m-1}A_1^* + A_1V_{l-1,m+1} + V_{l-1,m+1}A_{-1}^*),$$

$$m = \begin{cases} 2, 4, \ldots, l-2 & l\text{-even}, \\ 1, 3, \ldots, l-2 & l\text{-odd}. \end{cases}$$

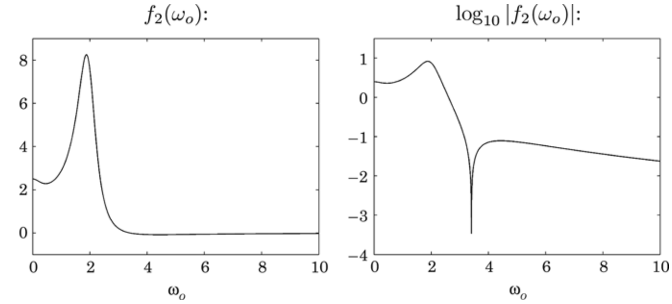

Fig. 2. Plots of $f_2(\omega_o)$ and $\log_{10}|f_2(\omega_o)|$ in the expression for the $\mathcal{H}_2$ norm of dissipative Mathieu equation with $a = 1$ and $b = 0.2$.

Application of Theorem 3 is illustrated in Section 6 on two examples: the dissipative Mathieu equation of Section 2.1 and the two-dimensional oscillating channel flow of Section 2.2.

6. Examples

In this section, we employ Theorem 3 to determine the second-order corrections to the $\mathcal{H}_2$ norms of systems described in Section 2.1 and Section 2.2.

6.1. The dissipative Mathieu equation

The $\mathcal{H}_2$ norm of the dissipative Mathieu equation subject to small-amplitude oscillations (see Section 2.1) is given by

$$\|\mathcal{H}\|_2^2 = f_0 + \epsilon^2 f_2(\omega_o) + O(\epsilon^4),$$

where $f_0 = 1/(4ab)$, and

$$f_2(\omega_o) = \frac{64ab^2 + 4\left(3a - 4b^2\right)\omega_o^2 - \omega_o^4}{2a^2b\left(4b^2 + \omega_o^2\right)\left(16a^2 - 8\left(a - 2b^2\right)\omega_o^2 + \omega_o^4\right)}.$$

The formula for $f_2(\omega_o)$ is obtained from Theorem 3 with the help of MATHEMATICA.

Plots of $f_2(\omega_o)$ and $\log_{10}|f_2(\omega_o)|$ in the expression for the $\mathcal{H}_2$ norm of the dissipative Mathieu equation with $a = 1$ and $b = 0.2$ are shown in Fig. 2. We observe two resonant peaks: the positive at $\omega_o \approx 1.88$, and the negative at $\omega_o \approx 3.40$. As can be seen from the plot of $\log_{10}|f_2(\omega_o)|$, the latter resonant peak has a very small magnitude compared to the peak at $\omega_o \approx 1.88$ and its contribution to the $\mathcal{H}_2$ norm is not likely to be significant.

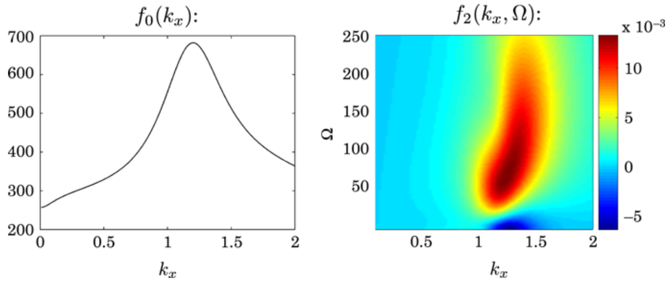

Fig. 3. Plots of $f_0(k_x)$ and $f_2(k_x, \Omega)$ in the expression for the $\mathcal{H}_2$ norm of two-dimensional oscillating channel flow with $R = 2000$.

6.2. *Two-dimensional oscillating channel flow*

The $\mathcal{H}_2$ norm of a two-dimensional oscillating channel flow is parameterized by the wave-number k_x, the Stokes number Ω, and the Reynolds numbers R and R_u (see Section 2.2). For small-amplitude oscillations of the lower wall ($R_u \ll R$), we use Theorem 3 to obtain

$$\left[\|\mathcal{H}\|_2^2\right](k_x) = f_0(k_x) + R_u^2 f_2(k_x, \Omega) + O(R_u^4),$$

where functions $f_0(k_x)$ and $f_2(k_x, \Omega)$ also depend on the Reynolds number R.

Fig. 3 illustrates plots of $f_0(k_x)$ and $f_2(k_x, \Omega)$ in the two-dimensional oscillating channel flow with $R = 2000$. These two functions are determined numerically using the Matlab Differentiation Matrix Suite (Weideman & Reddy, 2000) with 50 collocation points in the wall-normal direction. We observe a peak in the plot of $f_0(k_x)$ which is caused by 'poorly damped modes' of parallel channel flow $U_0(y)$. Clearly, depending on the value of Stokes number Ω this peak can be attenuated or amplified in the presence of wall oscillations. For small values of Ω (approximately $\Omega < 20$) 'the periodic feedback' leads to a reduction in the $\mathcal{H}_2$ norm, whereas for large values of Ω (approximately $20 < \Omega < 250$) it increases the $\mathcal{H}_2$ norm. Thus, the perturbation analysis facilitates identification of the Stokes numbers (i.e. the wall oscillation frequencies) that lead to amplification or attenuation (relative to $U_0(y)$) of background disturbances. Once the wall oscillation frequency is selected using perturbation analysis, the influence of the wall oscillation amplitude on the $\mathcal{H}_2$ norm can be studied using, for example, the truncation of bi-infinite operators in the harmonic Lyapunov equation or so-called 'approximate modeling approach' (Zhou et al., 2003). We note that our analysis provides a computationally efficient method for determination of the $\mathcal{H}_2$ norm of periodic systems subject to small-amplitude oscillations without resorting to either of these two approaches. The only approximation in our analysis arises due to discretization of channel flow system in the wall-normal direction. As far as temporal dynamics is concerned, our analysis is *exact*.

7. Concluding remarks

We use a perturbation analysis to develop an efficient method for computation of the $\mathcal{H}_2$ norm of LTP systems with small-amplitude oscillations. We show that, up to second order in the perturbation parameter, the $\mathcal{H}_2$ norm can be determined by solving a conveniently coupled system of Lyapunov and Sylvester equations whose structure is determined by the structure of unperturbed LTI system. For finite-dimensional systems, the size of these equations corresponds to the size of matrices in the original LTI system. For infinite-dimensional systems, these equations are operator-valued and are typically solved by resorting to finite-dimensional approximation of the underlying operators. In the channel flow example, this amounts to solving Lyapunov and Sylvester equations of the size determined by discretization in the wall-normal direction (typically less than 50). The developed procedure is suitable for identification of forcing frequency $\omega_o = 2\pi/T$ that leads to the largest $\mathcal{H}_2$ norm reduction/increase.

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Appendix. Stochastic interpretation of the $\mathcal{H}_2$ norm

In this section we give a stochastic interpretation to the definition of the $\mathcal{H}_2$ norm using the notion of *cyclostationary* processes (Gardner, 1990). A stochastic process v is called cyclostationary if its statistical properties change periodically in time. In particular the variance of v, $R^v(t,t) = \mathscr{E}\{v(t)v^*(t)\}$, is a periodic function of t. It can be shown that the output of an LTP system whose input is a stationary process is cyclostationary (Gardner, 1990).

We consider an *infinite-dimensional* LTP system. The development for finite-dimensional LTP systems is standard and thus omitted. Let H denote the spatio-temporal kernel function of the infinite-dimensional LTP system. Let d be a white noise random field in *both* the spatial and temporal directions, and let $\phi = Hd$. Then

$$R^\phi(y,y;t,t) = \mathscr{E}\{\phi(y,t)\phi^*(y,t)\}$$
$$= \int_0^\infty \int_0^\infty \int_{-1}^1 \int_{-1}^1 H(y,\eta;t,\tau)\mathscr{E}\{d(\eta,\tau)d^*(\chi,s)\}$$
$$H^*(y,\chi;t,s)\mathrm{d}\eta\,\mathrm{d}\chi\,\mathrm{d}\tau\,\mathrm{d}s$$
$$= \int_0^\infty \int_{-1}^1 H(y,\eta;t,\tau)H^*(y,\eta;t,\tau)\mathrm{d}\eta\,\mathrm{d}\tau,$$

where the last equality follows from the fact that d is a white noise spatio-temporal random field and therefore $R^d(y,\eta;t,s) = \mathscr{E}\{d(\eta,\chi)d^*(\eta,s)\} = I\delta(\eta-\chi)\delta(\tau-s)$, (VanMarcke, 1988). We have thus established that

$$\mathrm{tr}(R^\phi(y,y;t,t))$$
$$= \int_0^\infty \int_{-1}^1 \mathrm{tr}(H^*(y,\eta;t,\tau)H(y,\eta;t,\tau))\mathrm{d}\eta\,\mathrm{d}\tau.$$

Comparing the above equation to the definition of the $\mathcal{H}_2$ norm for infinite-dimensional systems we have

$$\|\mathcal{H}\|_2^2 = \frac{1}{T}\int_0^T \int_{-1}^1 \mathrm{tr}(R^\phi(y,y;t,t))\mathrm{d}y\,\mathrm{d}t. \tag{A.1}$$

Eq. (A.1) means that the $\mathcal{H}_2$ norm of a spatially distributed LTP system is equal to the average over one temporal period and sum over the entire spatial domain of the variance of the output random field, when the input random field is spatio-temporal white noise.

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