On the optimal design of structured feedback gains for interconnected systems

Makan Fardad, Fu Lin, and Mihailo R. Jovanović

***Abstract*— We consider the design of optimal static feedback gains for interconnected systems subject to architectural constraints on the distributed controller. These constraints are in the form of sparsity requirements for the feedback matrix, which means that each controller has access to information from only a limited number of subsystems. We derive necessary conditions for the optimality of structured static feedback gains in the form of coupled matrix equations. In general these equations have multiple solutions, each of which is a stationary point of the objective function. For stable open-loop systems, we show that in the limit of expensive control, the optimal controller can be found analytically using perturbation techniques. We use this feedback gain to initialize homotopy-based gradient and Newton iterations that find an optimal solution to the original (non-expensive) control problem. For unstable open-loop systems, the centralized truncated gain is used as an initial estimate for the iterative schemes aimed at finding the optimal structured feedback gain. We consider both spatially invariant and spatially varying problems and illustrate our developments with several examples.**

***Index Terms*— Architectural constraints, interconnected systems, optimal decentralized control, structured feedback gains.**

I. INTRODUCTION

Large-scale systems are ubiquitous in nature and in modern engineering applications. The individual components of a large-scale system are often equipped with sensing, computation, and actuation capabilities. A central question in the control of such systems then becomes, *what is a good communication architecture between the controllers of the different subsystems*? Clearly there is a trade-off in play: the best possible performance is attained if all controllers can communicate to each other, and as a whole decide upon the control action to be applied to each subsystem. This, however, comes at the expense of excessive communication and computation requirements. The other extreme is when every controller acts in isolation, and applies a control action to its corresponding subsystem based on its own measurement of the subsystem's output. This scenario places minimum communication and computation requirements on the controllers, but generally comes at the expense of poor performance.

A desired scenario, and a reasonable middle ground, is the *local* communication of subsystems with their immediate or nearest neighbors, referred to as localized control. It is this type of decentralized control that is the focus of the present research effort.

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The synthesis problem of distributed control for interconnected systems has received considerable attention in recent years [1]–[14]. For linear spatially invariant plants, it was shown in [1] that optimal controllers are themselves spatially invariant. Furthermore, for optimal distributed problems with quadratic performance indices the dependence of a controller on information coming from other parts of the system decays exponentially as one moves away from that controller [1]. These developments motivate the search for inherently *localized* controllers. For example, one could search for optimal controllers that are subject to the condition that they communicate only to other controllers within a certain radius. However, the framework of [1] does not allow the *a priori* specification of the communication architecture of the distributed controller. Additionally, it does not provide error bounds on the deviation from optimality if one were to truncate the information dependence of every controller, for example by confining it to communicating within a prespecified radius of itself.

Thus the problem of controller design for large-scale and distributed systems is mostly dominated by the architectural and localization constraints imposed on the controller. Such design problems are well-known to be difficult, with almost three decades of research in an area that has come to be known as the 'decentralized control of large-scale systems'; see [15] and references therein.

A problem of wide interest is to search for an optimal distributed controller that is a static gain (i.e., has no temporal dynamics) with *a priori* assigned localization constraints. Such controllers are less sophisticated than dynamic controllers, and thus may achieve lower performance, but have the advantage of being much easier to implement. Most architectural requirements on the distributed static controller are in the form of sparsity constraints. We focus particularly on cases where the static feedback gain can be partitioned into banded matrices, which are non-zero only on the main diagonal and a relatively small number of sub-diagonals. Such banded structure translates to each controller using only local information to compute the control action. We search for structured controllers that minimize the H_2 norm. We find a *coupled* set of algebraic matrix equations that characterize necessary conditions for the optimality of the structured static controller.

The challenging aspect of solving the aforementioned coupled equations is that they can have multiple solutions, each of which is a stationary point of the norm minimization problem. In general, it is not known how many local minima exist or how to find them. For stable open-loop systems, we consider the case of expensive control [16] in which a perturbation analysis of the necessary conditions for optimality can be performed. The expensive control scenario restricts the norm of the distributed gain as it tries to minimize the use of control effort. Perturbation analysis results in equations that are decoupled and thus readily solvable to find expansion terms of arbitrarily high-order. This leads to a unique optimal gain that is small in norm; this gain can be used to initialize a homotopy-based gradient and Newton iterations to determine optimal controller for smaller values of control penalty. In this numerical scheme the level of expensiveness of the control problem is successively reduced and the resulting matrix equations are solved until the original (non-expensive) control objective is recovered. For unstable open-loop systems, we apply a spatial truncation on the optimal centralized feedback gain to initialize an iterative numerical scheme that is aimed at solving necessary conditions for optimality subject to structural constraints. An interesting example of this procedure arises in the case of spatially invariant systems; for such systems the optimal static feedback is a centralized gain whose spatial dependence decays exponentially with distance [1]. We therefore expect that the truncated gain would be a reasonable initial estimate for the Newton-based iterative scheme that attempts to find the optimal structured feedback gain.

The paper is organized as follows: in Section II the structured optimal control problem is formulated and necessary conditions for optimality are found. In Section III we consider stable open-loop systems and apply a perturbation analysis on these necessary conditions to find the unique optimal expensive static controller. In Section IV gradient- and Newton-based iterative schemes are used to find stabilizing (locally) optimal distributed controllers. In Section V we provide illustrative examples, and in Section VI we summarize our developments.

II. Necessary conditions for optimality

Consider the control problem

$$\begin{aligned} \dot{\psi} &= A\,\psi + B_1\,w + B_2\,u, \\ z &= C_1\,\psi + D\,u, \\ y &= C_2\,\psi, \qquad\qquad u \;=\; -F\,y, \end{aligned}$$

where $C_1 = \begin{bmatrix} Q^{1/2} & 0 \end{bmatrix}^*$ and $D = \begin{bmatrix} 0 & R^{1/2} \end{bmatrix}^*$. The matrix F denotes the static feedback gain which is subject to structural constraints. Specifically, the structural constraints dictate the zero entries of the feedback gain. We assume that the subspace $\mathcal{S}$ encapsulates these constraints and that there is a stabilizing $F \in \mathcal{S}$.

Upon closing the loop, the above problem can equivalently be written as

$$\begin{aligned} \dot{\psi} &= (A - B_2 F C_2)\,\psi + B_1\,w, \\ z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2} F C_2 \end{bmatrix} \psi. \end{aligned} \qquad \text{(H2)}$$

Note that w denotes exogenous signals and that the performance output z encapsulates both the amplitude of the state and that of the control input. We now consider the following optimal control problem:

- *Find the matrix $F \in \mathcal{S}$ such that $\|H\|_2^2$ is minimized, where H is the transfer function from w to z and $\|\cdot\|_2$ is the H_2 norm.*

It can be shown [17] that this optimal control problem is equivalent to

$$\begin{aligned} &\text{minimize} && J \;=\; \text{trace}\,(P B_1 B_1^*) \\ &\text{subject to} && (A - B_2 F C_2)^*\,P \,+\, P\,(A - B_2 F C_2) \\ &&& = -\,(Q \,+\, C_2^* F^* R F C_2)\,, \;\; F \in \mathcal{S}. \end{aligned} \qquad \text{(SG)}$$

Note that the first constraint in (SG) is nothing but the Lyapunov equation $A_{\text{cl}}^*\,P + P\,A_{\text{cl}} = -C_{\text{cl}}^* C_{\text{cl}}$ corresponding to the closed-loop system (H2). We remark that – when there are no structural constraints on F – the above problem is strongly related to the static output-feedback LQR problem considered in [18].

Most structural requirements on the gain F are in the form of *sparsity* constraints. A sparse matrix is populated primarily with zeros and the pattern of the non-zero entries describes the communication architecture of the distributed controller; if the ijth block of F is non-zero it means that subsystem j is communicating its state to the controller of the ith subsystem. We focus particularly on cases where F is a banded matrix, i.e., it is only non-zero on its main block-diagonal and a relatively small number of block sub-diagonals. For vehicular platoons, such a banded structure translates to each vehicle using only information about the position and velocity of a relatively small number of neighboring vehicles to control its own position and velocity.

Theorem 1 (Necessary conditions for optimality): In order for matrix $F \in \mathcal{S}$, with $A - B_2 F C_2$ Hurwitz, to be optimal for the problem (SG) it is necessary that it satisfies the following set of equations:

$$\begin{aligned} &(A - B_2 F C_2)^*\,P \,+\, P\,(A - B_2 F C_2) \\ &\qquad = -\,(Q \,+\, C_2^* F^* R F C_2)\,, && \text{(NC1)} \\ &(A - B_2 F C_2)\,L + L\,(A - B_2 F C_2)^* \;=\; -B_1 B_1^*, && \text{(NC2)} \\ &(R F C_2 L C_2^*) \circ I_{\mathcal{S}} \;=\; (B_2^* P L C_2^*) \circ I_{\mathcal{S}}, && \text{(NC3)} \end{aligned}$$

where $\circ$ denotes the element-wise multiplication of matrices and the matrix $I_{\mathcal{S}}$ is defined as

$$[I_{\mathcal{S}}]_{ij} = \begin{cases} 1 & \text{if } F_{ij} \text{ is a free variable,} \\ 0 & \text{if } F_{ij} = 0 \text{ is required.} \end{cases}$$

Remark 1: Similar necessary conditions for optimality of fixed-order dynamic controllers appear in [19]. To differentiate our results from those of [19], we note that the equivalent of equation (NC3) in [19] is given in the form of summation of matrix multiplications, which is much more expensive to compute than (NC3). Further, for stable open-loop systems we develop homotopy-based descent method by utilizing the perturbation analysis in the limit of expensive control. Also, the Newton direction for the design of static structured feedback gains is determined for the first time in our work.

We consider (NC3) for a few important examples of $\mathcal{S}$:

- *$\mathcal{S}$ is the subspace of diagonal matrices.* In this case $I_{\mathcal{S}} = I$. The optimality condition (NC3) becomes
$$(R F C_2 L C_2^*) \circ I \;=\; (B_2^* P L C_2^*) \circ I,$$
which can also be written as $\text{diag}\,\{R F C_2 L C_2^*\} = \text{diag}\,\{B_2^* P L C_2^*\}$.
- *$\mathcal{S}$ is the subspace of tridiagonal matrices.* In this case $I_{\mathcal{S}} = T$, where T is the tridiagonal matrix with elements equal to one on the main diagonal, first upper, and first lower subdiagonals. The optimality condition (NC3)

becomes

$$(RFC_2LC_2^*) \circ T \;=\; (B_2^*PLC_2^*) \circ T.$$

- *$\mathcal{S}$ is the subspace of $N \times 2N$ matrices with tridiagonal and diagonal $N \times N$ blocks.* This scenario would arise, for example, in vehicular platoons where the measurement matrix C_2 is a $2N \times N$ matrix and its $N \times N$ blocks correspond to position and velocity measurements. The block corresponding to the velocity is diagonal if every vehicle has access to its own velocity measurements. The block corresponding to position is tridiagonal if every vehicle has access to its own position and positions of its immediate neighbors. In this case $I_{\mathcal{S}} = \begin{bmatrix} T & I \end{bmatrix}$, and the optimality condition (NC3) becomes

$$(RFC_2LC_2^*) \circ \begin{bmatrix} T & I \end{bmatrix} \;=\; (B_2^*PLC_2^*) \circ \begin{bmatrix} T & I \end{bmatrix}.$$

  Note that both T and I in this equation have dimension $N \times N$. An alternative way of writing this equation is as follows. Define

$$G \;=\; RFC_2LC_2^* \,-\, B_2^*PLC_2^* \;=\; \begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

  where G_1, G_2 are $N \times N$ partitions of G. Then the optimality condition (NC3) can be written as

$$G_1 \circ T = 0, \quad G_2 \circ I = 0.$$

Example 1 (mass-spring system): Consider a mass-spring system shown in Fig. 1. If restoring forces are considered as linear functions of displacements, the dynamics of the system with unit masses and spring constants are given by

$$A \;=\; \begin{bmatrix} O & I \\ A_{21} & O \end{bmatrix}, \;\; B_1 \;=\; B_2 \;=\; \begin{bmatrix} O \\ I \end{bmatrix},$$

where I and O are, respectively, $N \times N$ identity and zero matrices, and $A_{21} := \mathrm{toeplitz}\,([\,-2 \;1\; 0 \;\cdots\; 0\,])$. We consider a situation in which the control applied on the nth mass has access to: (i) displacement and velocity of the nth mass, and (ii) displacements of the p neighboring masses. Thus, the output matrix C_2 is the $2N \times 2N$ identity matrix and $I_{\mathcal{S}} = [\; S_p \;\; I \;]$, with S_p being the identity matrix for $p = 0$, a tridiagonal matrix of ones for $p = 1$, a pentadiagonal matrix of ones for $p = 2$, and so on.

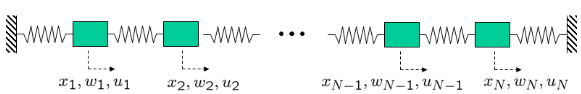


Fig. 1. Mass-spring system.

Consider now the problem (SG) where matrix C_2 is a banded matrix that contains the subsystems' communication architecture, and F is a *diagonal* matrix. Let $R = r\,I$ where $r > 0$. Then it is easy to show that the necessary conditions (NC1)-(NC3) given in Theorem 1 simplify to

$$\begin{aligned} &(A - B_2FC_2)^*\,P \,+\, P\,(A - B_2FC_2) \\ &\qquad\qquad = \,-\,(Q \,+\, C_2^*F^*RFC_2)\,, \\ &(A - B_2FC_2)\,L \,+\, L\,(A - B_2FC_2)^* \;=\; -B_1B_1^*, \\ &F \;=\; (1/r)\,([B_2^*PLC_2^*] \circ I)\,([C_2LC_2^*] \circ I)^{-1}. \end{aligned} \qquad \text{(NC)}$$

Condition (NC) consists of two Lyapunov equations in P and L that are *coupled* together by the final equation.

The optimal structured static controller can thus be found numerically by solving the coupled set of algebraic matrix equations (NC). The challenging aspect of this problem lies in the fact that these equations can have multiple solutions, each of which is a stationary point of the objective function J. In general, it is not known how many such local minima exist or how to find them. This difficulty persists even in the unstructured problems, as pointed out by [18]. In Section III we demonstrate how these issues are circumvented in the framework of expensive control.

A. *Spatially invariant systems*

If system (H2) is spatially invariant, then the application of an appropriate Fourier transform leads to a family of systems parameterized by the spatial frequency variable θ [1]

$$\begin{aligned} \dot{\psi}_\theta \;&=\; (A_\theta - B_{2\theta}F_\theta C_{2\theta})\,\psi_\theta \,+\, B_{1\theta}\,w_\theta, \\ z_\theta \;&=\; \begin{bmatrix} Q_\theta^{1/2} \\ -R_\theta^{1/2}F_\theta C_{2\theta} \end{bmatrix}\psi_\theta. \end{aligned} \qquad \text{(H2')}$$

For systems defined over discrete spatial lattice $\mathbb{Z}$, the level of spatial spread in (H2') is determined by the presence of the terms $\mathrm{e}^{\pm \mathrm{j} p\theta}$ in the state-space parameters. Note that $\mathrm{j} = \sqrt{-1}$, $\theta \in [0,\, 2\pi)$, and $p \in \mathbb{N}$; higher values of p indicate larger spatial spread in (H2'). For example, the presence of $\mathrm{e}^{\pm \mathrm{j}\theta}$ (corresponding to $p = 1$) in A_θ indicates communication with first immediate neighbors in the updating of the state variables of each subsystem.

For spatially invariant systems, the optimal control problem (H2') can always be formulated in such a way that matrix $C_{2\theta}$ contains the subsystems' communication architecture, and F is *independent* of θ. In particular, for $R = rI$ the necessary conditions for optimality (NC) further simplify to:

$$\begin{aligned} &(A_\theta - B_{2\theta}FC_{2\theta})^*\,P_\theta \,+\, P_\theta\,(A_\theta - B_{2\theta}FC_{2\theta}) \\ &\qquad\qquad = \,-\,(Q_\theta \,+\, C_{2\theta}^*F^*RFC_{2\theta})\,, \\ &(A_\theta - B_{2\theta}FC_{2\theta})\,L_\theta \,+\, L_\theta\,(A_\theta - B_{2\theta}FC_{2\theta})^* \\ &\qquad\qquad = \,-\,B_{1\theta}B_{1\theta}^*, \\ &F \;=\; (1/r)\,(\int_\theta B_{2\theta}^*P_\theta L_\theta C_{2\theta}^*\,\mathrm{d}\theta)\,(\int_\theta C_{2\theta}L_\theta C_{2\theta}^*\,\mathrm{d}\theta)^{-1}. \end{aligned} \qquad \text{(NC')}$$

We make the following important observations:

- There are no structural constraints on the matrix F in (NC') as long as it provides the closed-loop stability; i.e., as long as $A_\theta - B_{2\theta}FC_{2\theta}$ is Hurwitz for each θ.
- The dimension of matrices in equations (NC') is significantly smaller than that in equations (NC). In fact, the 'scale' of the problem is fully absorbed by the spatial transform variable θ. This constitutes the main advantage of applying (spatial) Fourier methods to spatially invariant systems; as shown in [1], even for infinite-dimensional systems, Fourier transform techniques render analysis and design problems to those for a θ-parameterized family of finite dimensional systems.
- For systems defined over discrete spatial lattice $\mathbb{Z}_N = \{0, \ldots, N-1\}$, the integration in the last equation of (NC') should be interpreted as summation over the frequency variable $\theta \in \{0, \ldots, N-1\}$.

Example 2 (system of single integrator vehicles on circle): Consider a system of identical vehicles over an undirected discrete circle $\mathbb{Z}_N$, where the derivative of the scalar state of each vehicle is to be determined by the weighted difference of the state of that vehicle and the average of the states of its two immediate neighbors. In this case $\theta \in \{0, \ldots, N-1\}$ and the optimal scalar feedback gain, $F = f_1 = \text{const.}$, for the control problem (H2') with $\{A_\theta = 0,\, B_{1\theta} = B_{2\theta} = 1,\, C_{2\theta} = 2\,(1 - \cos(2\pi\theta/N)),$ $R_\theta = r,\, Q_\theta \geq 0\}$ can be determined analytically and is given by $f_1 = \left((1/(r\sum_{\theta=1}^{N-1} C_{2\theta}))\sum_{\theta=1}^{N-1}(Q_\theta/C_{2\theta})\right)^{1/2}$.

III. Perturbation analysis of expensive control

In this section we demonstrate that for stable open-loop systems an expensive control scenario allows for a significant reduction in the complexity of problem (SG).

Consider again (SG) where the open-loop system is assumed to be stable (i.e., A is a Hurwitz matrix) and, for simplicity, F is restricted to being diagonal

$$\begin{array}{ll} \text{minimize} & J(F) \,=\, \operatorname{trace}(PB_1B_1^*) \\ \text{subject to} & (A - B_2FC_2)^*\,P \,+\, P\,(A - B_2FC_2) \\ & = -\,(Q \,+\, C_2^*F^*RFC_2)\,,\ F \text{ is diagonal.} \end{array}$$

The difficulty in solving this optimization problem is that it is not convex in F. We therefore consider a simpler problem in which $R = (1/\varepsilon)\,I$, with $0 < \varepsilon \ll 1$; we will henceforth refer to this as *expensive optimal control problem*. Then, by representing P, L, and F as

$$P = \sum_{n=0}^{\infty}\varepsilon^n P^{(n)},\ L = \sum_{n=0}^{\infty}\varepsilon^n L^{(n)},\ F = \sum_{n=1}^{\infty}\varepsilon^n F^{(n)},$$

substituting in (NC), and employing perturbation analysis, we obtain the set of *conveniently coupled* equations given by (EXP). Note that these equations are only coupled in one direction, in the sense that for any $n \geq 1$ the $O(\varepsilon^n)$ equations depend only on the solutions of the $O(\varepsilon^{n-1})$ equations and are not coupled among themselves. Thus the perturbation expansion terms can be readily computed up to any order.

Matrix F found by this procedure is the *unique optimal* (in the sense of perturbations) solution of the expensive optimal control problem. This is due to the fact that the equations (EXP), under the assumption of convergence, give a unique matrix $F = \sum_{n=1}^{\infty}\varepsilon^n F^{(n)}$. In fact, the radius of convergence of the perturbation parameter ε of the above series expansions can be established in terms of the open-loop system parameters.

IV. Algorithms for computation of structured feedback gains

A. Gradient and Newton directions

We employ the standard iterative descent algorithms to solve the optimization problem (SG). Specifically, given an initial stabilizing feedback gain $F_0 \in \mathcal{S}$, the iterative algorithm generates a minimizing sequence $\{F_k \in \mathcal{S}\}$ as follows

$$F_{k+1} \,=\, F_k \,+\, s_k\mathcal{F}_d(F_k),$$

where $\mathcal{F}_d(F_k)$ is a descent direction of the objective function J evaluated at F_k, and s_k is the step-size. In this section, we consider two descent directions: negative gradient and Newton directions. We determine step-size using the standard backtracking line search to guarantee the closed-loop stability and a decrease in the value of J. The iterative algorithms for computation of structured feedback gains are then presented in Section IV-B.

The computational cost and the convergence rate of an iterative algorithm depend heavily on its descent direction. Generally speaking [20], gradient method is computationally cheaper than the Newton's method but the convergence rate is linear. On the other hand, Newton's method is computationally expensive but the convergence rate is quadratic. Additional details about iterative algorithms and descent directions can be found in [20], [21].

We next summarize the results for gradient- and Newton-based iterations; detailed derivations are omitted due to space limitation and will be reported elsewhere. The gradient direction of the objective function J in (SG) is given by

$$\mathcal{F}_g \,=\, \nabla J(F) \,=\, 2(RFC_2LC_2^* \,-\, B_2^*PLC_2^*)\circ I_{\mathcal{S}}, \quad (1)$$

where P and L are, respectively, determined by (NC1) and (NC2). Specifically, given F, the Lyapunov equations (NC1) and (NC2) are solved for P and L; the gradient direction is then obtained by substituting these into (1).

If the optimization variable is a *vector* as considered in standard optimization literature (e.g., [20], [21]), then the Newton direction is given by

$$\mathcal{F}_{nt} \,=\, -(\nabla^2 J(F))^{-1}\,\nabla J(F),$$

where $\nabla^2 J(F)$ is the *Hessian* of the objective function J. In this case, $\nabla J(F)$ is a vector and $\nabla^2 J(F)$ is a square invertible matrix. However, the optimization variable in our case is a *structured matrix* which necessitates generalization of the formula for $\mathcal{F}_{nt}$. In this case, the Newton direction represents the solution of the following linear equation

$$\mathcal{H}(\tilde{F}) \,+\, \nabla J(F) \,=\, 0,$$

where $\mathcal{H}$ is a linear operator representing the generalized Hessian for matricial optimization variables. In particular, the Newton direction $\mathcal{F}_{nt}$ of the objective function J is the solution of the following linear equation in $\tilde{F}$ with $\tilde{F} \in \mathcal{S}$

$$-\,\mathcal{F}_g \,=\, 2(R\tilde{F}C_2L \,-\, GZ_1 \,-\, B_2^*Z_2L)C_2^* \circ I_{\mathcal{S}}, \quad (2)$$

where $G = RFC_2 - B_2^*P$ and Z_1, Z_2 represent the solutions to the following Lyapunov equations

$$\begin{array}{l} A_{\text{cl}}\,Z_1 \,+\, Z_1\,A_{\text{cl}}^* \,=\, -\,(B_2\tilde{F}C_2L \,+\, LC_2^*\tilde{F}^*B_2^*), \\ A_{\text{cl}}^*\,Z_2 \,+\, Z_2\,A_{\text{cl}} \,=\, -\,(C_2^*\tilde{F}^*G \,+\, G^*\tilde{F}C_2), \end{array} \quad (3)$$

respectively. Note that the unknown variables are Z_1, Z_2 and $\tilde{F}$. Specifically, given F, one first computes the gradient direction $\mathcal{F}_g$ and then solves the system of linear equations (2,3) for $\mathcal{F}_{nt} = \tilde{F}$.

It is noteworthy that the equations for the necessary conditions for optimality (NC1)-(NC3) are *nonlinear* in variable F and that the equations for the Newton direction are *linear* in variable $\tilde{F}$. To solve for $\tilde{F}$, one approach is to vectorize $\tilde{F}$ using the Kronecker product [22] and to solve the resulting linear equation.

B. Implementation

It is assumed that the initial structured feedback gain F_0 gives a stable closed-loop system. With $\mathcal{F}_d$ given by

$$
\begin{aligned}
O(1):& \quad F^{(0)} \,=\, 0 \\
O(\varepsilon):& \left\{ \begin{aligned} A^*P^{(0)} \,+\, P^{(0)}A \,&=\, -Q \\ AL^{(0)} \,+\, L^{(0)}A^* \,&=\, -B_1B_1^* \\ F^{(1)} \,&=\, ([B_2^*P^{(0)}L^{(0)}C_2^*] \circ I)\,([C_2L^{(0)}C_2^*] \circ I)^{-1} \end{aligned} \right. \\
O(\varepsilon^2):& \left\{ \begin{aligned} A^*P^{(1)} \,+\, P^{(1)}A \,&=\, (B_2F^{(1)}C_2)^*\,P^{(0)} \,+\, P^{(0)}\,(B_2F^{(1)}C_2) \,-\, C_2^*F^{(1)*}F^{(1)}C_2 \\ AL^{(1)} \,+\, L^{(1)}A^* \,&=\, (B_2F^{(1)}C_2)\,L^{(0)} \,+\, L^{(0)}\,(B_2F^{(1)}C_2)^* \\ F^{(2)} \,&=\, ([B_2^*P^{(0)}L^{(1)}C_2^* \,+\, B_2^*P^{(1)}L^{(0)}C_2^* \,-\, F^{(1)}C_2L^{(1)}C_2^*] \circ I)\,([C_2L^{(0)}C_2^*] \circ I)^{-1} \end{aligned} \right. \\
&\;\;\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned} \tag{EXP}
$$

either the negative gradient direction, $\mathcal{F}_d = -\mathcal{F}_g$, or the Newton direction, $\mathcal{F}_d = \mathcal{F}_{nt}$, we next employ the standard backtracking line search, with parameters $\alpha = 0.3$ and $\beta = 0.5$ (see [21, p. 464]), to determine the step-size. In addition to providing decrease in J, it is also necessary to guarantee the closed-loop stability when selecting the step-size. Hence, given descent direction $\mathcal{F}_d$ and step-size $s = 1$:

Stepsize search

repeat: $s := \beta s$

until: both conditions are satisfied

1) $J(F + s\mathcal{F}_d) < J(F) + \alpha\, s\, \mathrm{trace}(\nabla J(F)^T \mathcal{F}_d)$;
2) $A_{\mathrm{cl}} = A - B_2(F + s\mathcal{F}_d)C_2$ is Hurwitz.

Iterative algorithm

Given a stabilizing F_0, the iterative algorithm at each step k is given by

1) compute descent direction $\mathcal{F}_d(F_k)$;
2) use step-size search to determine s_k;
3) update $F_{k+1} = F_k + s_k\mathcal{F}_d(F_k)$.

until: stopping criterion $\|\nabla J(F_k)\|_2 < \epsilon$ is reached.

C. Homotopy-based iterations

For stable open-loop systems, the design procedure of Section III can be extended to non-expensive control regimes via the use of *homotopy* (i.e., *continuation*) methods. Homotopy methods are based on replacing the problem of interest by a *parameterized family* of problems with following properties: (i) the problem is easily solved for small values of the parameter; and (ii) as the parameter value is increased the problem transforms into the problem of interest.

We thus consider the following iterative scheme to find the optimal structured F that solves optimization problem (SG). Let $R = (1/\varepsilon)\, I$, where ε is a positive scalar and I is the identity matrix. Treating ε as the homotopy parameter, we first find the optimal structured F that minimizes (SG) for very small values of ε. This has the interpretation of expensive control [16]; a small ε results in a large R, which in light of the output matrix in (H2) means that control effort has to be spent sparingly. The solution of this problem results in a structured feedback gain F that is small in norm and is computed easily and reliably using MATLAB (for very small ε, the matrix F can be computed using the perturbation analysis of Section III). We now slightly increase the value of ε and use the obtained F to initialize the next round of gradient or Newton iterations. We continue increasing the value of ε until the matrix R of the original control objective is recovered. This algorithm effectively *guides* the optimization scheme through the many local minima.

V. Examples

A. Spatially varying mass-spring system

In this section we revisit a mass-spring system on a line described in Example 1. The centralized optimal controller, $u = -K\psi$, is given by $K = R^{-1}B_2^*X$, where X is the positive definite solution of the algebraic Riccati equation

$$A^*X \,+\, XA \,+\, Q \,-\, XB_2R^{-1}B_2^*X \,=\, 0.$$

The initial feedback gain F_0 is determined by projecting K onto the information structure $\mathcal{S}$, that is

$$F_0 \,=\, [\, F_{0p} \;\; F_{0v} \,] \,=\, K \circ [\, S_p \;\; I \,].$$

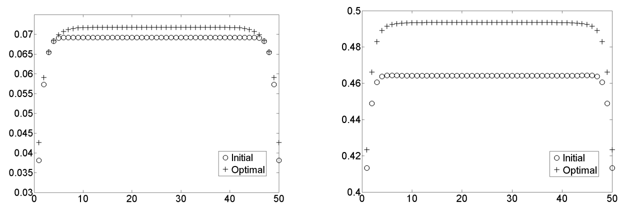
It turns out that F_0 for any $p \geq 0$ is a stabilizing feedback gain. Thus, the iterative descent algorithm of Section IV-A can be initialized with F_0. We note that other approaches have been developed to find a stabilizing feedback gain [7]. The stopping criterion for the algorithms is $\|\nabla J(F)\|_2 < 10^{-6}$. For system with $N = 50$ masses, $Q = I$, and $R = 10I$, the change in initial and optimal values, J_0 and $J^\star$, with respect to the spatial spread p is reported in Table I. The optimal value of J with the centralized feedback gain K is $J_c = 230.7099$. For smaller values of spatial spread p, the iterative algorithms of Section IV-A effectively enhance performance relative to truncated centralized controller. As p increases, performance improvement becomes less significant; this indicates that nearly optimal performance can be achieved with truncated centralized controllers of large-enough spatial spread.

TABLE I

Initial and optimal values J_0 and $J^\star$ with respect to p for mass-spring system with $N = 50$, $Q = I$, and $R = 10I$.

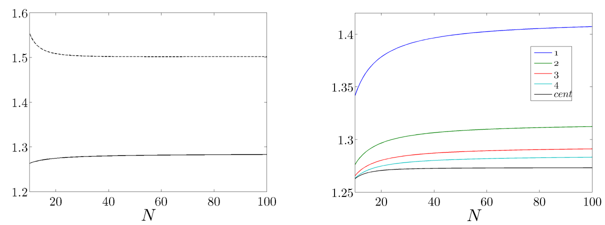
	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$
J_0	270.2621	247.2561	243.6723	242.7422	242.4594
$J^\star$	248.6063	244.4403	242.8970	242.3327	242.1396

Comparisons between elements on the main diagonals of $(F_{0p}, F_p^\star)$ and $(F_{0v}, F_v^\star)$ for mass-spring system with $\{p = 1,\, N = 50,\, Q = I,\, R = 10I\}$ are given in Fig. 2(a) and Fig. 2(b), respectively.



(a) (b)

Fig. 2. Comparison of elements on the main diagonals of: (a) initial and optimal position feedback gains; (b) initial and optimal velocity feedback gains. Mass-spring system of Example 1 with $\{p = 1,\ N = 50,\ Q = I,\ R = 10I\}$ is considered.



(a) (b)

Fig. 3. Scaling of $\|\bar{H}\|_2^2 := (1/N)\|H\|_2^2$ with the number of vehicles N for: (a) truncated optimal centralized controller (dashed) and structured controller (solid) with spatial spread $p = 4$; (b) structured (with spatial spread $p = \{1, 2, 3, 4\}$) and centralized state-feedback controllers.

B. System of single integrator vehicles on circle

We next consider the design of structured controllers of spatial spread p for a system of identical single integrator vehicles described in Example 2 with $C_{2\theta} = [\, 2\,(1 - \cos{(2\pi\theta/N)}) \ \cdots \ 2\,(1 - \cos{(2\pi p\,\theta/N)})\,]^*$, $F = [\, f_1 \ \cdots \ f_p \,]$, and the following pair of state/control weights $(Q_\theta, r) = (2\,(1 - \cos{(2\pi\theta/N)})\,, 1)$. The selected state penalty accounts for the difference between the position of vehicle n and the average of positions of two neighboring vehicles. In this case, the optimal centralized state-feedback controller, $u_\theta = -K_\theta \psi_\theta$, is given by $K_\theta = Q_\theta^{1/2}$, and the optimal structured controller with $p = 1$ is given in Example 2. To determine the structured controller with $p \geq 2$, we apply Newton's method to solving (NC') by initiating the iterations with a truncated centralized controller. Figure 3(a) illustrates the performance comparison of the truncated centralized controller and the structured controller with spatial spread $p = 4$; clearly, Newton's iteration provides appreciable performance improvement relative to the truncated centralized controller. The performance of structured controllers with spatial spread $p = \{1, 2, 3, 4\}$ (obtained by applying Newton's method to (NC')) and optimal centralized controller is shown in Fig. 3(b). As expected, an increase in the spatial spread decreases the performance gap between optimal structured and optimal centralized designs.

VI. Conclusions

We study the static H_2 synthesis problem subject to the feedback gain satisfying certain sparsity constraints. These constraints are such that they enforce a *localized* communication architecture between the underlying plant and the controller. We find necessary conditions for optimality of localized controllers that are in the form of coupled matrix equations. In general these equations have multiple solutions, which correspond to different stationary points. However, for stable open-loop systems, in the limit of expensive control, perturbation methods can be used to find a unique optimal controller. We use this controller to initialize a homotopy-based numerical optimization scheme that determines optimal controller in the non-expensive regime. For unstable open-loop systems, an iterative scheme utilizing gradient and Newton's methods is employed to determine a solution to stationary conditions for optimality. The iterations are initialized by projecting optimal centralized controllers to the set of controllers with desired localization properties. The presented algorithms appear to work well in practice as demonstrated by illustrative examples in Section V.

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