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On Optimal Link Removals for Controllability Degradation in Dynamical Networks

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***Abstract*— We consider the problem of controllability degradation in dynamical networks subjected to malicious attacks. Attacks on the networks are assumed to be in the form of the removal of interconnection links. We formulate an optimization problem that seeks sets of links whose removal causes maximal degradation in the rank of the controllability gramian. We apply the alternating direction method of multipliers and sequential convex programming to find local solutions to this combinatorial optimization problem. We provide illustrative examples to demonstrate the utility of our results.**

***Index Terms*— Alternating direction method of multipliers (ADMM), cyber-physical systems, link failure, rank minimization, sequential convex programming.**

I. Introduction

Investigating the robustness of networks to targeted and random attacks is a problem of significant interest across various scientific communities. Understanding robustness properties will allow the identification of vulnerabilities and the design of mitigation strategies against attacks, will provide insight on the connection between robustness and network topology, and will aid the design of networks that are resilient to cyber attacks [1], [2]. Most of the existing literature in this area has focused on static networks with relatively simple or no network dynamics. Furthermore, measures of robustness, such as network connectivity and average length of shortest paths, are also geared towards static networks.

In the case of static networks, important high level conclusions have been drawn from studies that connect network topology to its robustness. For example, scale-free networks have been shown to be more robust to random node removals compared to exponential networks [3]. Since most real-world networks are composed of interconnected dynamical systems, it is of interest to develop similar robustness results for dynamical networks.

In this paper we develop an optimization-based framework for analyzing the robustness properties of dynamical networks against link removal attacks. We consider the problem of identifying small sets of links whose removal will lead to maximal controllability degradation of the network. The classical notions of controllability and observability have received attention lately in the context of complex network [4]–[6], where in particular, the problems of finding the minimum number of control inputs and observation nodes were studied for large-scale networks. The impact of stochastic link failures on synchronization in large-scale dynamical networks was studied in [7]. In the context of cyber security of power networks, [8]–[10] address the robustness properties of state estimators to malicious attacks. A novel metric that combines the controllability and observability properties of dynamical networks for vulnerability analysis was developed in [11].

We employ the optimization framework developed in [12]–[14] to formulate the link removal problem. However, our work here is different from [12]–[14] in that, rather than minimizing a system norm, we search for links whose failure minimizes the *rank* of the controllability (or observability) gramian. The latter problem is combinatorial and intractable, in general. We employ the alternating direction method of multipliers [15], together with an iterative algorithm based on sequential convex programming [16], to solve the optimization problem. We use numerical examples to demonstrate the utility of our results.

The paper is organized as follows. In Section II we present the problem formulation. An equivalent formulation, which better lends itself to convex programming, is presented in Section III. We apply the alternating direction method of multipliers and sequential convex programming to find local solutions in Section IV. Numerical results are presented in Section V, followed by conclusion and future directions in Section VI.

II. Problem Statement & Formulation

We consider the following discrete-time linear time invariant model for the network

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k),$$

where $x(k) \in \mathbb{R}^n$ is the state of the network at time k, $u(k) \in \mathbb{R}^m$ is the exogenous input, and $y(k) \in \mathbb{R}^p$ is the output measurement. The system matrix A models the interaction among network nodes. We assume that the network is under attack, and that the attacker has access to the interconnection links between nodes. The attacker seeks to remove a few 'critical links' whose failure would result in maximal degradation of the system's controllability or observability. In the rest of this paper we will focus only on the degradation of controllability; the formulation for observability degradation follows readily from a duality argument.

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Under the attack of a malicious agent, the A-matrix of the system changes to $A_f = A + \Delta$, where the effect of Δ is to 'zero out' certain (non-diagonal and originally nonzero) entries of the matrix A; this models the removal of a link between two nodes. More precisely, we consider $\Delta = -A \circ F$, where F is a matrix with entries in $\{0, 1\}$ and $\circ$ denotes *elementwise* matrix multiplication. For example, suppose the matrix F is zero everywhere except at its ijth entry and $F_{ij} = 1$. This means that the attacker has dismantled the link between nodes i and j. Therefore $A_f = A - A \circ F$ is a matrix that is identical to A everywhere except at its ijth entry, and $(A_f)_{ij} = 0$.

In general, we may wish to further restrict the class of matrices in which F is allowed to vary, by constraining it to a set $\mathcal{S}$. The set $\mathcal{S}$ describes which entries of F can be nonzero and which can not. For instance, if we know that certain links of the system are completely out of reach of attackers, then the set $\mathcal{S}$ will only include matrices that have zeros in the entries corresponding to those links, i.e., those links can never be made to fail by a malicious agent. Also, the nonzero entries of F allowed by $\mathcal{S}$ are a subset of the nonzero entries of A, since it is not possible to attack a link that does not exist.

Throughout this work, we assume that link failures do not result in system instability. For a stable system defined by the pair $(A - A \circ F, B)$, the controllability gramian $X \succeq 0$ is found by solving the algebraic Lyapunov equation

$$X - (A - A \circ F)X(A - A \circ F)^T = BB^T.$$

We formulate the optimal link failure problem that causes maximal degradation of controllability as

$$\begin{array}{ll} \text{minimize} & \operatorname{rank}(X) \\ \text{subject to} & X - (A - A \circ F)X(A - A \circ F)^T = BB^T \\ & X \succeq 0, \ F \in \mathcal{S}, \ \mathbb{1}^T F \, \mathbb{1} = \kappa \\ & F_{ij} \in \{0, 1\}, \ i, j = 1, \ldots, n, \end{array} \tag{1}$$

where the optimization variables are the matrices F and X, $\mathbb{1}$ denotes the column vector of all ones, and κ is a positive integer that denotes the number of failed links. Since the entries of F take values in $\{0, 1\}$ the term $\mathbb{1}^T F \, \mathbb{1}$ counts the number of unit elements in F.

The above optimization problem is intractable in general due to (a) the nonconvexity of the rank function, (b) the nonlinearity of the matrix equality constraint, and (c) the binary constraints on the entries of F. In the following section, we propose an equivalent reformulation of (1) that better lends itself to the application of convex optimization algorithms.

For simplicity of notation, we hereafter summarize the binary constraints on the entries of F as

$$F_{ij} \in \{0, 1\},$$

with the understanding that this holds for all $i, j = 1, \ldots, n$. Furthermore, we replace $F \in \mathcal{S}$ in (1) with the equivalent constraint

$$S \circ F = 0, \tag{2}$$

where S is a matrix of the same dimension as F and defined as

$$S_{ij} = \begin{cases} 1 & \text{if } F_{ij} = 0 \text{ is required} \\ 0 & \text{if } F_{ij} \text{ is a free variable.} \end{cases}$$

For example, if the diagonal entries of A are out of reach of attackers (implying that links internal to every node can not be removed or made to fail) then F will be a matrix whose diagonal entries are constrained to zero, and (2) becomes $I \circ F = 0$.

III. An Equivalent Formulation

In this section we reformulate (1) in a way that is more amenable to relaxations and convex programing. The following proposition is one of the main results of this work.

Proposition 1: Assuming that $A - A \circ F$ is a stable matrix, the optimization problem (1) is equivalent to

$$\begin{array}{ll} \text{minimize} & \operatorname{trace}(X) \\ \text{subject to} & \begin{bmatrix} X - BB^T & (A - A \circ F)X \\ X(A - A \circ F)^T & X \end{bmatrix} \succeq 0 \\ & S \circ F = 0, \ \mathbb{1}^T F \, \mathbb{1} = \kappa, \ F_{ij} \in \{0, 1\}, \end{array} \tag{3}$$

where the optimization variables are the matrices F and X, and the binary constraints on the entries of F are required for all $i, j = 1, \ldots, n$.

Proof of Proposition 1: We begin the proof by demonstrating that problem (1) is equivalent to the optimization problem

$$\begin{array}{ll} \text{minimize} & \operatorname{rank}(X) + \varepsilon \operatorname{trace}(X) \\ \text{subject to} & X - (A - A \circ F)X(A - A \circ F)^T \succeq BB^T \\ & X \succeq 0, \ S \circ F = 0, \ \mathbb{1}^T F \, \mathbb{1} = \kappa, \ F_{ij} \in \{0, 1\}, \end{array} \tag{4}$$

for any $\varepsilon > 0$. Problem (4) is the same as (1) except for the replacement of the Lyapunov equation in (1) with a Lyapunov *inequality* and the addition of $\varepsilon \operatorname{trace}(X)$ to the objective.

Let $\mathcal{L}$ denote the Lyapunov operator $\mathcal{L}(X) := X - (A - A \circ F)X(A - A \circ F)^T$, defined for stable $A - A \circ F$. The inverse Lyapunov operator is monotonically increasing, in the sense that if $Q_1 \succeq Q_2$ then $X_1 := \mathcal{L}^{-1}(Q_1) \succeq X_2 := \mathcal{L}^{-1}(Q_2)$. To see this, let Q_1 and Q_2 be positive semidefinite matrices such that $Q_1 \succeq Q_2 \succeq 0$. Then

$$(A - A \circ F)^k Q_1 (A - A \circ F)^{kT} \succeq (A - A \circ F)^k Q_2 (A - A \circ F)^{kT}$$

for $k = 0, 1, \ldots$, and therefore [17]

$$\begin{aligned} X_1 &:= \sum_{k=0}^{\infty} (A - A \circ F)^k Q_1 (A - A \circ F)^{kT} \\ &\succeq \sum_{k=0}^{\infty} (A - A \circ F)^k Q_2 (A - A \circ F)^{kT} =: X_2 \end{aligned}$$

if $A - A\circ F$ is stable. Recalling that the solution of the Lyapunov equation $\mathcal{L}(X) = Q$ for $Q \succeq 0$ and stable $A - A\circ F$ is given by $X = \sum_{k=0}^{\infty}(A - A\circ F)^k Q (A - A\circ F)^{kT} \succeq 0$, we have thus shown that $X_1 = \mathcal{L}^{-1}(Q_1) \succeq X_2 = \mathcal{L}^{-1}(Q_2)$ for $Q_1 \succeq Q_2$, with equality holding if and only if $Q_1 = Q_2$.

Now suppose X_1 and X_2 satisfy the Lyapunov inequality and Lyapunov equation in (4) and (1), respectively. Then the inequality $X_1 - (A - A\circ F)X_1(A - A\circ F)^T \succeq BB^T$ can be regarded as the matrix equality

$$X_1 - (A - A\circ F)X_1(A - A\circ F)^T = BB^T + E$$

for some $E \succeq 0$. Letting $Q_1 = BB^T + E$ and $Q_2 = BB^T$, from the argument in the previous paragraph it follows that $X_1 \succeq X_2$, with equality holding if and only if $E = 0$. In particular, this implies $\operatorname{rank}(X_1) \geq \operatorname{rank}(X_2)$ and $\operatorname{trace}(X_1) \geq \operatorname{trace}(X_2)$, with $\operatorname{trace}(X_1) = \operatorname{trace}(X_2)$ if and only if $X_1 = X_2$.

For fixed F, let the positive semidefinite matrix X^* satisfy the Lyapunov equation in (1). Then there exist infinitely many $\check{X} \succeq X^*$ that satisfy the Lyapunov inequality in (4) and for which $\operatorname{rank}(\check{X}) = \operatorname{rank}(X^*)$. However, from the previous paragraph it follows that the $\check{X}^*$ which satisfies the Lyapunov inequality and additionally minimizes $\operatorname{rank}(\check{X}) + \varepsilon \operatorname{trace}(\check{X})$ is unique and satisfies $\check{X}^* = X^*$. We conclude that, for fixed F, any $\check{X}^*$ that minimizes the objective $\operatorname{rank}(\check{X}) + \varepsilon \operatorname{trace}(\check{X})$ subject to the Lyapunov inequality in (4) necessarily satisfies

$$\check{X}^* - (A - A\circ F)\check{X}^*(A - A\circ F)^T = BB^T.$$

Problems (1) and (4) are thus equivalent.

We now invoke [18, Thm. II.2], which implies that for fixed F the minimization of $\operatorname{trace}(X)$ subject to a linear matrix inequality in the positive semidefinite matrix X results in an optimal X that is of minimal rank. This indicates that the minimization of $\operatorname{trace}(X)$ subject to the matrix inequality in (4) is equivalent to the minimization of any conic combination of $\operatorname{rank}(X)$ and $\operatorname{trace}(X)$, and in particular the minimization of $\operatorname{rank}(X) + \varepsilon \operatorname{trace}(X)$, subject to the same constraint. Thus, problem (4) is equivalent to the optimization problem

$$\begin{array}{ll} \text{minimize} & \operatorname{trace}(X) \\ \text{subject to} & X - (A - A\circ F)X(A - A\circ F)^T \succeq BB^T \\ & X \succeq 0,\ S\circ F = 0,\ \mathbb{1}^T F \mathbb{1} = \kappa,\ F_{ij} \in \{0,1\}. \end{array} \tag{5}$$

Problem (5) is the same as problem (4) except that $\operatorname{rank}(X) + \varepsilon \operatorname{trace}(X)$ in (4) has been replaced by $\operatorname{trace}(X)$.

Finally, using the identity $X = XX^\dagger X$ in which $X^\dagger$ denotes the pseudo-inverse of X, we rewrite the Lyapunov inequality in (5) as

$$X - (A - A\circ F)XX^\dagger X(A - A\circ F)^T \succeq BB^T.$$

Applying a Schur complement [19, p. 28] to this matrix inequality yields

$$\begin{bmatrix} X - BB^T & (A - A\circ F)X \\ X(A - A\circ F)^T & X \end{bmatrix} \succeq 0,$$

which is the same as the matrix inequality in (3). The proof of the proposition is now complete. ■

IV. Solving (3) via ADMM

Problem (3) is still nonconvex due to the binary constraints on the entries of F and the bilinear terms $(A - A\circ F)X$ appearing in the matrix inequality. In this section we use the alternating direction method of multipliers (ADMM) [15], which is well-suited for problems with binary constraints, and sequential convex programming [16] to find local solutions to (3).

We start by endowing (3) with the redundant elementwise matrix inequality constraint $0 \leq F \leq \mathbb{1}\mathbb{1}^T$ to obtain

$$\begin{array}{ll} \text{minimize} & \operatorname{trace}(X) \\ \text{subject to} & \begin{bmatrix} X - BB^T & (A - A\circ F)X \\ X(A - A\circ F)^T & X \end{bmatrix} \succeq 0 \\ & S\circ F = 0,\ 0 \leq F \leq \mathbb{1}\mathbb{1}^T \\ & F_{ij} \in \{0,1\},\ \mathbb{1}^T F \mathbb{1} = \kappa. \end{array} \tag{6}$$

Let $f(F, X)$ be defined as

$$f(F,X) = \begin{cases} \operatorname{trace}(X) & \text{if } (F,X) \text{ satisfies 1st, 2nd, and 3rd constraints in (6)} \\ \infty & \text{otherwise,} \end{cases}$$

and $g(F)$ defined as [13]

$$g(F) = \begin{cases} 0 & \text{if } F_{ij} \in \{0,1\} \text{ and } \mathbb{1}^T F \mathbb{1} = \kappa \\ \infty & \text{otherwise.} \end{cases}$$

Problem (6) can thus be equivalently expressed as

$$\text{minimize} \quad f(F,X) + g(F).$$

To rewrite this optimization problem in a form that lends itself to the application of ADMM [15] we introduce the auxiliary variable K,

$$\begin{array}{ll} \text{minimize} & f(F,X) + g(K) \\ \text{subject to} & F - K = 0. \end{array} \tag{7}$$

The augmented Lagrangian corresponding to this optimization problem is given by

$$L_\rho(F,X,K,\Lambda) = f(F,X) + g(K) + \operatorname{trace}(\Lambda^T[F-K]) + (\rho/2)\|F-K\|_F^2.$$

ADMM now finds a local solution to (7) by iteratively executing the following steps for $l = 0, 1, \ldots$,

$$(F^{l+1}, X^{l+1}) := \arg\min_{F,X} L_\rho(F,X,K^l,\Lambda^l), \tag{8}$$

$$K^{l+1} := \arg\min_{K} L_\rho(F^{l+1},X^{l+1},K,\Lambda^l), \tag{9}$$

$$\Lambda^{l+1} := \Lambda^l + \rho\,(F^{l+1} - K^{l+1}), \tag{10}$$

until both of the conditions $\|F^{l+1} - K^{l+1}\|_F \le \epsilon$ and $\|K^{l+1} - K^l\|_F \le \epsilon$ are satisfied. For simplicity, we refer to (8) and (9) as F-minimization and K-minimization problems, respectively.

It is not difficult to show that the F- and K-minimization steps result in the respective optimization problems

$$\begin{array}{ll} \text{minimize} & \operatorname{trace}(X) + (\rho/2)\|F - \hat{K}^l\|_F^2 \\ \text{subject to} & \begin{bmatrix} X - BB^T & (A - A\circ F)X \\ X(A - A\circ F)^T & X \end{bmatrix} \succeq 0 \\ & S\circ F = 0,\ 0 \le F \le \mathbb{1}\mathbb{1}^T, \end{array} \tag{11}$$

and

$$\begin{array}{ll} \text{minimize} & (\rho/2)\|K - \hat{F}^l\|_F^2 \\ \text{subject to} & K_{ij} \in \{0,1\},\ \mathbb{1}^T K\, \mathbb{1} = \kappa, \end{array} \tag{12}$$

where $\hat{K}^l := K^l - (1/\rho)\Lambda^l$ and $\hat{F}^l := F^{l+1} + (1/\rho)\Lambda^l$. The important observation here is that problem (12), which has absorbed all of the binary constraints, can be solved *analytically*. Indeed, (12) has the closed-form solution [13]

$$K_{ij} = \begin{cases} 1 & \text{if } (\hat{F}^l)_{ij} \ge [\hat{F}^l]_\kappa \\ 0 & \text{if } (\hat{F}^l)_{ij} < [\hat{F}^l]_\kappa, \end{cases}$$

where $[\hat{F}^l]_\kappa$ denotes the κth largest entry of $\hat{F}^l$. In other words, to find the solution of (12) we identify the location of the κ largest entries of $\hat{F}^l$ and set the corresponding entries of K equal to one. It now remains to solve the F-minimization problem (11), which we elaborate on in the rest of this section.

Solving F-minimization Problem (11) via Sequential Convex Programming

In what follows, we use sequential convex programming [16] to iteratively approximate (11) with a convex program. We begin by rewriting (11) as

$$\begin{array}{ll} \text{minimize} & \operatorname{trace}(X) + (\rho/2)\|F - \hat{K}^l\|_F^2 + \lambda \operatorname{trace}(Z_-) \\ \text{subject to} & \begin{bmatrix} X - BB^T & (A - A\circ F)X \\ X(A - A\circ F)^T & X \end{bmatrix} = Z_+ - Z_- \\ & S\circ F = 0,\ 0 \le F \le \mathbb{1}\mathbb{1}^T,\ Z_+ \succeq 0,\ Z_- \succeq 0. \end{array} \tag{13}$$

For large enough λ, the minimizer of (13) is equal to that of (11), [16, pp. 14-15]. To see this, let us denote by Z the matrix on the left of the inequality constraint in (11). The matrix Z is set equal to $Z_+ - Z_-$ in (13), where both Z_+ and Z_- are positive semidefinite. Now, rather than enforcing $Z \succeq 0$ as in (11), we allow Z to contain a negative semidefinite component $-Z_-$, but penalize Z_- in the objective. As λ grows, the matrix Z_- approaches zero and the original matrix inequality in (11) is recovered.

We now formulate a convex approximation of (13) by linearizing the terms $(A - A\circ F)X$ and $X(A - A\circ F)^T$ around our current best estimate $(\mathcal{X}, \mathcal{F})$ of (X, F). We thus formulate

$$\begin{array}{ll} \text{minimize} & \operatorname{trace}(X) + (\rho/2)\|F - \hat{K}^l\|_F^2 + \lambda \operatorname{trace}(Z_-) \\ \text{subject to} & \begin{bmatrix} X - BB^T & H(X,F) \\ H(X,F)^T & X \end{bmatrix} = Z_+ - Z_- \\ & S\circ F = 0,\ 0 \le F \le \mathbb{1}\mathbb{1}^T,\ Z_+ \succeq 0,\ Z_- \succeq 0, \\ & \|X - \mathcal{X}\|_F^2 \le r^2,\ \|F - \mathcal{F}\|_F^2 \le r^2, \end{array} \tag{14}$$

where

$$H(X,F) := (A - A\circ F)\mathcal{X} + (A - A\circ\mathcal{F})X - (A - A\circ\mathcal{F})\mathcal{X}.$$

The function H is a linearization of $(A - A\circ F)X$ around $(\mathcal{X}, \mathcal{F})$, obtained by replacing X, F with $\mathcal{X} + \delta X$, $\mathcal{F} + \delta F$, where $\|\mathcal{X}\| \gg \|\delta X\|$, $\|\mathcal{F}\| \gg \|\delta F\|$, writing a Taylor expansion, eliminating high order terms in δX, δF, and substituting $X - \mathcal{X}$, $F - \mathcal{F}$ for δX, δF, respectively. The last two inequalities in (14) use the Frobenius norm to restrict the search to a trust region [16, p. 15] with radius r around the current best estimate $(\mathcal{X}, \mathcal{F})$; these inequalities can be equivalently replaced by the constraints

$$\begin{bmatrix} W & X - \mathcal{X} \\ (X - \mathcal{X})^T & I \end{bmatrix} \succeq 0, \quad \operatorname{trace}(W) \le r^2,$$

$$\begin{bmatrix} V & F - \mathcal{F} \\ (F - \mathcal{F})^T & I \end{bmatrix} \succeq 0, \quad \operatorname{trace}(V) \le r^2.$$

Finally, replacing $\|F - \hat{K}^l\|_F^2$ in the objective with $\operatorname{trace}(Y)$ and adding the additional constraint

$$\begin{bmatrix} Y & F - \hat{K}^l \\ (F - \hat{K}^l)^T & I \end{bmatrix} \succeq 0,$$

would allow problem (14) to be reformulated as an SDP.

The convex problem (14) is solved as part of an iteration. The iterative procedure is initialized at $\mathcal{F}_0 := 0$ and $\mathcal{X}_0 := X_\mathrm{c}$, where X_c is the controllability gramian of the original system

$$X_\mathrm{c} - AX_\mathrm{c}A^T = BB^T.$$

We summarize this in Algorithm 1.

Algorithm 1 Iterative algorithm for local solutions to (11)

1: **given** ρ, $\hat{K}^l$, S, $\lambda \gg 1$, r, and ε.
2: **for** $i = 1, 2, \ldots$ **do**
3: If $i = 1$, set $\mathcal{X} := X_\mathrm{c}$, $\mathcal{F} := 0$.
4: If $i > 1$, set $\mathcal{X}$, $\mathcal{F}$ equal to optimal X, F from previous iteration.
5: Solve (14) to obtain X^*, F^*.
6: If $\|X^* - \mathcal{X}\| < \varepsilon$, $\|F^* - \mathcal{F}\| < \varepsilon$, **quit**.
7: **end for**

If Algorithm 1 converges for small enough values of ε and large enough values of λ, then $H(X^*, F^*) = (A - A\circ F^*)X^*$ and $Z_-^* = 0$. We make no claim on the convergence of Algorithm 1 or the global optimality of the solution that results from it. However, in our numerical experiments Algorithm 1 always converges; see Sec. V for examples.

V. Examples

We briefly explain the conventions we use for visualizing link removals in directed graphs, for all examples in this section. Arrowheads are employed to demonstrate the direction of links. Dashed links are those identified for removal by the optimization algorithm. Regular nodes are represented by small dots, whereas larger boxes indicate nodes through which exogenous inputs enter the network. In all examples we take the set $\mathcal{S}$, and the corresponding matrix S, to be such that only the links corresponding to nonzero and non-diagonal entries of the A-matrix are subject to failure. We seek the solution of (3) using the ADMM algorithm described in Sec. IV with parameters $\rho = 1$ and $\varepsilon = 5 \times 10^{-6}$.

Example 1: We consider a network with 5 nodes, an all-to-all interconnection topology, a randomly-generated A-matrix, and two independent inputs such that the system is controllable. Figs. 1(a), (b), (c) respectively demonstrate computational results for $\kappa = 4, 6, 8$. The outcomes for $\kappa = 4, 6$ are not surprising, as the optimization algorithm selects to remove links emanating from the input nodes, thus attempting to isolate these nodes from the rest of the network. For $\kappa = 8$ we observe that the algorithm additionally selects links that are distant from the input nodes and the removal of which does not further decrease the rank of the contrallability gramian. We believe this to be a result of the approximation introduced by linearization in the optimization algorithm.

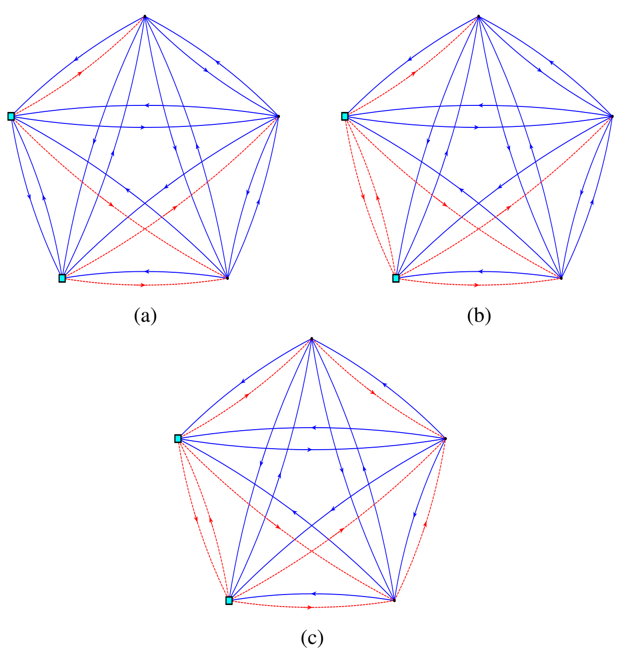

Fig. 1: Networks for Example 1. Dashed links are those selected by the optimization algorithm for removal. Figures (a), (b), (c) respectively correspond to $\kappa = 4, 6, 8$.

Example 2: We consider a network with two clusters as in Fig. 2, interacting only through a pair of directed links, and assume that the input is injected into the cluster on the left and ensure that the network is controllable. Figs. 2(a), (b), (c) respectively demonstrate computational results for $\kappa = 1, 3, 5$. For $\kappa = 1$ the algorithm selects a link originating from the input node, the failure of which does not lead to a decrease in the rank of the controllability gramian. It is important to note that the globally optimal solution in this case is the link that goes from the left cluster to the right cluster, the failure of which would render the entire right cluster uncontrollable. When the number of attacked links is increased to $\kappa = 3$, the optimization algorithm removes all links emanating from the input node. This leaves the input node as the only controllable node in the network and thus the rank of the gramian drops to to one. Not surprisingly, further increase in the value of κ to 5 has no effect on the rank of the gramian.

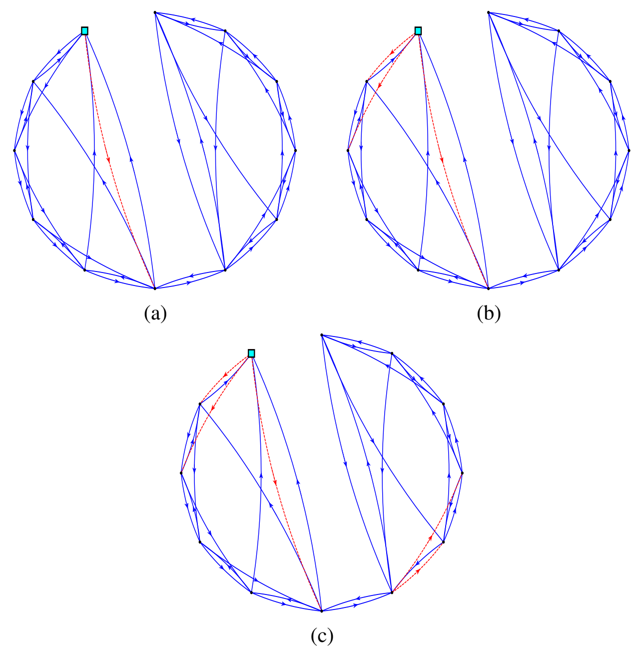

Fig. 2: Networks for Example 2. Dashed links are those selected by the optimization algorithm for removal. Figures (a), (b), (c) respectively correspond to $\kappa = 1, 3, 5$.

Example 3: We consider a small-world network [20] of 15 nodes, constructed from a regular graph with probability $p = 0.1$ of link rewiring. To choose the input nodes with the highest control authority, we proceed with a greedy method as follows. We inject an input at a single node and compute the corresponding controllability gramian and its rank. We repeat this for all 15 nodes and order them from highest to lowest rank. For networks whose gramians have the same rank, ordering is performed according to the condition number of the gramians. Having obtained an ordered list of input-node candidates, we now consider two scenarios: in the first scenario only the best node is taken as the input node, and in the second scenario the best five nodes are taken as input nodes. In both cases we choose $\kappa = 10$ and implement the optimal link selection algorithm. The links removed for the network with one input and five inputs respectively are shown in Fig. 3 and Fig. 4. It is interesting to note that, when the network has a single input, the links chosen to be removed are not all close to the input node. On the other hand, when the network has five inputs, the links removed are chosen close to the input nodes.

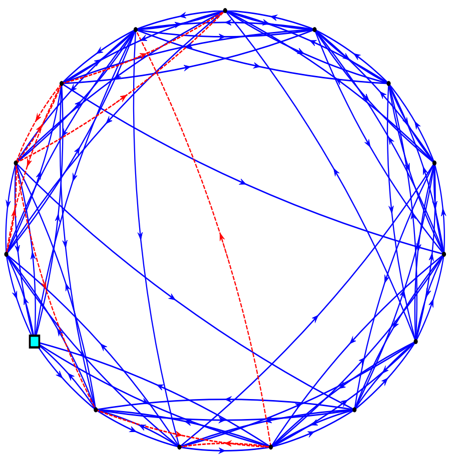

Fig. 3: Network for Example 3 with one input.

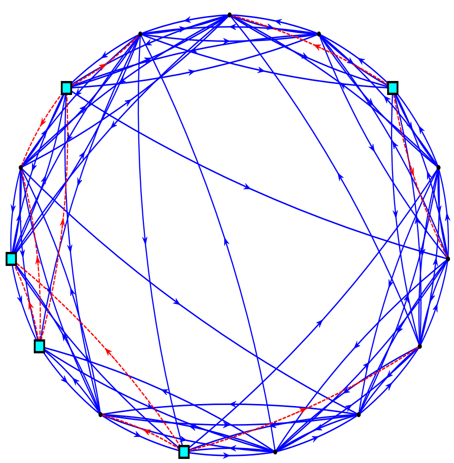

Fig. 4: Network for Example 3 with five inputs.

VI. Conclusions and Future Work

Motivated by the desire to identify vulnerabilities in large-scale interconnected systems, we present an optimization-based framework for the identification of critical links whose removal leads to maximal controllability degradation. Due to the nonconvexity and combinatorial nature of the optimal link failure problem, we employ approximations and iterative algorithms to find its local solutions.

As our numerical experiments demonstrate, the proposed optimization algorithm is not always successful at finding the globally optimal solution. Thus, part of our future research efforts will be to find better convex relaxations of the optimal link failure problem. Our future work will also focus on the application of the proposed framework to networks in which nodes, each described by a dynamical system, are coupled via specific interconnection topologies such as scale-free, exponential, random, and small-world. In particular, it is of great interest to compare the robustness properties of such dynamical networks with corresponding results reported in the literature for the case of static networks. We expect to draw conclusions that help illuminate the interplay between the internal dynamics of nodes and the network topology, and use this towards the design of resilient networks.

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