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PERTURBATION METHODS IN STABILITY AND NORM ANALYSIS OF SPATIALLY PERIODIC SYSTEMS*

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Abstract. We consider systems governed by partial differential equations with spatially periodic coefficients over unbounded domains. These spatially periodic systems are considered as perturbations of spatially invariant ones, and we develop perturbation methods to study their stability and $\mathcal{H}^2$ system norm. The operator Lyapunov equations characterizing the $\mathcal{H}^2$ norm are studied by using a special frequency representation, and formulas are given for the perturbation expansion of their solution. The structure of these equations allows for a recursive method of solving for the expansion terms. Our analysis provides conditions that capture possible resonances between the periodic coefficients and the spatially invariant part of the system. These conditions can be regarded as useful guidelines when spatially periodic coefficients are to be designed to increase or decrease the $\mathcal{H}^2$ norm of a spatially distributed system. The developed perturbation framework also gives simple conditions for checking whether a spatially periodic operator generates a holomorphic C_0 semigroup and thus satisfies the spectrum-determined growth condition.

Key words. PDE with periodic coefficients, perturbation analysis, $\mathcal{H}^2$ norm, sectorial operator, spectrum-determined growth condition

AMS subject classifications. 35B27, 35B34, 35B35, 93C20, 93C73, 93D20, 93D25

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1. Introduction.

The terms distributed-parameter and infinite-dimensional are used to describe those systems in which the state belongs to an infinite-dimensional vector space [1]. Such systems include, but are not limited to, time-delay (retarded) and spatially distributed systems [2]. The latter includes systems in which the dynamics are governed by partial differential equations (PDEs) and it is a subclass of these systems that will be the subject of this study. More specifically, we will analyze certain properties of spatially distributed systems in which the underlying PDEs have spatially periodic coefficients. We refer to such systems as *spatially periodic*. Spatially periodic systems have many real life applications, for example, in boundary layer and channel flow problems with corrugated walls and in nonlinear optics.

Our motivation for this work is to study the effect of spatially periodic coefficients on stability and system norms of spatially distributed systems. This can be thought of as using the periodic coefficients as static feedback controls for spatially distributed systems. For example, in flow control problems where corrugated wall geometries or spatially periodic body forces are introduced, the PDEs that describe the resulting flow dynamics have periodic coefficients that are related to either the wall shape or the spatially distributed body force. An important objective is to “design” such wall shapes or body forces to obtain certain stability or instability properties of the resulting dynamics. There are currently no systematic methods for achieving this.

An analogy can be made between the present work and the use of time-periodic coefficients in ordinary differential equations (ODEs). It is known that the introduction

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of time-periodic coefficients in ODEs with constant coefficients can change the stability properties of the linear time invariant (LTI) system described by the original ODE. A useful picture is to think of an ODE with periodic coefficients as an LTI system modified by time-periodic (memoryless) feedback. It is known that certain unstable LTI systems can be stabilized by being put in feedback with periodic gains of properly designed amplitudes and frequencies [3]. This can be roughly considered as an example of "vibrational control" [4]. On the other hand, certain stable or neutrally stable LTI systems can be destabilized by periodic feedback gains. This is sometimes referred to as "parametric resonance" in the dynamical systems literature [3]. In the above scenarios, the stabilization/destabilization process depends in subtle ways on "resonances" between the natural modes of the LTI subsystem and the frequency and amplitude of the periodic feedback. Although Floquet analysis can be used to ascertain stability of the resulting periodic systems, it is cumbersome to use for *designing* the periodic coefficients. This requires an exhaustive search over frequencies and amplitudes of the periodic coefficients. Alternatively, simple resonance conditions can be derived by using a perturbation analysis [3], which in turn can be used for a preliminary selection of the coefficient's frequency. In this way, perturbation analysis serves as a useful design tool.

In related recent work [5] we developed computational tools to study stability and system norms for spatially periodic systems. However, for problems where the spatially periodic coefficients are to be designed, using these tools involves a computationally expensive search over spatial frequencies and amplitudes of the coefficients. Therefore, our aim in the present work is to develop a *perturbation analysis* that can be used to derive resonance conditions and provide a useful design tool in a similar manner to the case of ODEs discussed earlier. These resonance conditions can then identify candidate spatial frequencies to be used for the periodic coefficients. The exact behavior with respect to amplitudes can then be ascertained by using the full analysis of [5]. In this manner we reduce the dimension of the search space required for design problems.

Another challenging problem is checking the stability of a spatially periodic or, in general, any infinite-dimensional system. It is well known that, for a finite-dimensional LTI system, the spectrum of the infinitesimal generator (i.e., the A-matrix) being contained in the open left half of the complex plane is equivalent to exponential stability. In this sense the spectrum of the infinitesimal generator determines stability. Therefore it is said that the system satisfies the *spectrum-determined growth condition* (SDGC). But the SDGC may not hold for some infinite-dimensional LTI systems; indeed the evolution can grow exponentially even though the infinitesimal generator (i.e., the A-operator) has its spectrum inside the left half of the complex plane [6, 7, 8]. In the present work we use perturbation analysis to find simple conditions under which the spatially periodic system satisfies the SDGC and is exponentially stable.

Our presentation is organized as follows. Section 2 outlines the main results of the paper. Section 3 briefly reviews the frequency representation of spatially periodic operators. Section 4 introduces the problem setup. Section 5 discusses the analytic perturbation of the $\mathcal{H}^2$ norm, and section 6 provides related illustrative examples. Section 7 studies conditions under which a spatially periodic system is exponentially stable. Many proofs and technical details have been relegated to the appendix to improve readability.

Notation. We use $k \in \mathbb{R}$ to characterize the spatial-frequency variable, also known as the *wave number*. $\Sigma(T)$ is the spectrum of the operator T, $\Sigma_{\mathrm{p}}(T)$ its point spectrum, and $\rho(T)$ its resolvent set. $\mathbb{C}^-$ denotes all complex numbers with real part less than zero, and $j := \sqrt{-1}$. "*" denotes the complex-conjugate transpose and also the adjoint of a linear operator. $\overline{\mathfrak{S}}$ is the closure of the set $\mathfrak{S} \subset \mathbb{C}$. The function $\hat{u}(t,k)$ denotes the Fourier transform (in the spatial variable x) of the spatiotemporal function $u(t,x)$. Similarly, the operator $\hat{A}$ is the Fourier domain representation of the spatial operator A. We will use the same symbol for a linear operator and its kernel representation. Where there is no chance of confusion, we use the same notation for a spatially invariant operator and its Fourier symbol.

Terminology. Throughout the paper, we use the terms *spatial "operators"* and *spatial "systems."* By the former we mean a *purely spatial* operator with no temporal dynamics (i.e., a memoryless operator that acts on a spatial function and yields a spatial function), whereas by the latter we refer to a *spatiotemporal* system (a system with an internal state which evolves on some spatial domain; i.e., for every time t the state is a function on a spatial domain). By a *scalar system*, we mean that the Euclidean dimension of the state is equal to one. When spatially periodic feedback operators are small in norm, we will use the phrases *periodic feedback* and *periodic perturbation* interchangeably. Finally, in the case of doubly infinite matrices we will use the phrases biinfinite matrix and (biinfinite or lifted) operator interchangeably.

2. Main results.

We consider systems described by linear, time-invariant, integro partial differential equations defined on an unbounded one-dimensional domain. We use a standard state-space representation of the form

$$\begin{aligned} [\partial_t \psi](t,x) &= [A\,\psi](t,x) \;+\; [B\,u](t,x), \\ y(t,x) &= [C\,\psi](t,x), \end{aligned} \tag{1}$$

where $t \in [0,\infty)$ and $x \in \mathbb{R}$, ψ, u, y are spatiotemporal functions, and A, B, C are spatial integrodifferential operators with coefficients that are periodic functions with a common period X. We refer to (1) as a *spatially periodic system.*

It is often physically meaningful to regard the spatially periodic operators as additive or multiplicative perturbations of spatially invariant ones (and by spatially invariant we mean integrodifferential operators with constant coefficients). For example, the infinitesimal generator A in (1) can often be decomposed as $A = A^{\mathrm{o}} + \epsilon\,E$, where A^{o} is a spatially invariant operator and E is an operator that includes multiplication by periodic functions. In some control application the operator E is something to be "designed." Therefore it is desirable to have easily verifiable conditions for stability and norms of such systems. This would then allow for the selection of the spatial period and amplitude of periodic functions in E to achieve the desired behavior. The perturbation analysis we present, though limited to small values of ϵ, provides useful guidelines for selecting candidate "periods" for E.

Our results are derived by using a special frequency representation. We show that the spatial periodicity of the operators A, B, and C implies that (1) can be rewritten as

$$\begin{aligned} [\partial_t \psi_\theta](t) &= [\mathcal{A}_\theta\,\psi_\theta](t) \;+\; [\mathcal{B}_\theta\,u_\theta](t), \\ y_\theta(t) &= [\mathcal{C}_\theta\,\psi_\theta](t), \end{aligned} \tag{2}$$

where $\theta \in [0, 2\pi/X)$. For every value of θ, ψ_θ, u_θ, y_θ are *biinfinite vectors*, and $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, $\mathcal{C}_\theta$ are *biinfinite matrices.* The systems (2) and (1) are related through a unitary transformation, and in particular quadratic forms are preserved by this transformation. Consequently, the stability and quadratic norm properties of (2) and (1) are equivalent. With this transformation the analysis of the original system (1) is reduced to that of the family of systems (2) that are *decoupled* in the parameter θ. In particular, perturbation analysis for (2) is easier and less technical than for the original system (1).

To make for easier reading, both in this section and in the body of the paper, we first present the results on perturbation analysis of the $\mathcal{H}^2$ norm and then deal with the issue of stability.

The first set of results concerns $\mathcal{H}^2$ norm analysis. For a large class of infinite-dimensional systems, computing the $\mathcal{H}^2$ norm involves solving an *operator* algebraic Lyapunov equation

$$A\,P \,+\, P\,A^* \;=\; -B\,B^*.$$

In general this is a difficult task that must be done by using appropriate discretization techniques. However, if A and B are spatially periodic operators, then so is the solution P. Then the special frequency representation described above allows this operator Lypunov equation to be rewritten as a θ-decoupled family of Lyapunov equations

$$\mathcal{A}_\theta\,\mathcal{P}_\theta \,+\, \mathcal{P}_\theta\,\mathcal{A}_\theta^* \;=\; -\mathcal{B}_\theta\,\mathcal{B}_\theta^*, \tag{3}$$

where $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{P}_\theta$ are the *biinfinite matrix* representations of A, B, and P. Once $\mathcal{P}_\theta$ is found, the $\mathcal{H}^2$ norm of the system can be computed from [5]

$$\frac{1}{2\pi}\int_0^\Omega \operatorname{trace}[\mathcal{C}_\theta \mathcal{P}_\theta\,\mathcal{C}_\theta^*]\,d\theta, \qquad \Omega = 2\pi/X.$$

Solving (3) is still a difficult problem in general, since it involves biinfinite matrices. We use perturbation analysis as follows: The infinitesimal generator $\mathcal{A}_\theta$ is expressed as $\mathcal{A}_\theta = \mathcal{A}_\theta^{\mathrm{o}} + \epsilon\,\mathcal{E}_\theta$, where $\mathcal{A}_\theta^{\mathrm{o}}$ and $\mathcal{E}_\theta$ correspond to the spatially invariant and spatially periodic components of $\mathcal{A}_\theta$, respectively. It follows that the solution $\mathcal{P}_\theta$ is analytic in ϵ, and the terms of its power series expansion $\mathcal{P}_\theta^{(i)}$ satisfy a sequence of forward coupled Lyapunov equations. Furthermore, the terms $\mathcal{P}_\theta^{(i)}$ are banded matrices with the number of bands increasing with the index i. These Lyapunov equations can then be solved recursively for $i = 0, 1, 2, \ldots$. Formulas for these Lyapunov equations are derived in section 5. In some examples that we present in section 6, these formulas reveal simple "resonance" conditions for stabilization or destabilization of PDEs by using spatially periodic feedback.

The second set of results concerns the problem of stability. As mentioned in the introduction, when the infinitesimal generator A is an infinite-dimensional operator it is possible that its spectrum $\Sigma(A)$ lies inside $\mathbb{C}^-$ and yet $\|e^{At}\|$ grows exponentially [6, 7, 8]. In such cases it is said that the *spectrum-determined growth condition* is not satisfied [8]. Yet there exists a wide range of infinite-dimensional systems for which the spectrum-determined growth condition *is* satisfied. These include, but are not limited to, systems for which the infinitesimal generator is *sectorial* (also known as an operator which generates a *holomorphic* or *analytic* semigroup) [9, 10, 11] or the infinitesimal generator is a *Riesz-spectral* operator [2]. In this paper we focus on sectorial operators.

Therefore, to establish exponential stability of a system, one possibility would be to show simultaneously that

(i) the operator A is sectorial (and thus its spectrum determines stability) and
(ii) the spectrum $\Sigma(A)$ lies in $\mathbb{C}^-$.

The problem is that proving an infinite-dimensional operator is sectorial and finding its spectrum can in general be extremely difficult.

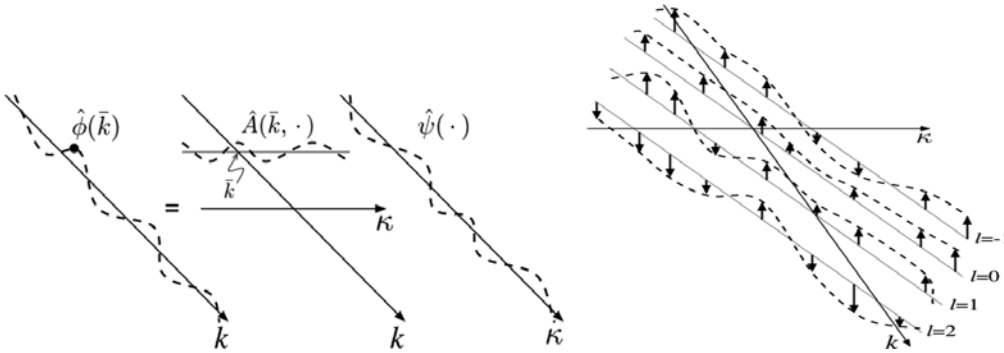

FIG. 1. *Left: Pictorial representation of* $\hat{\phi}(\bar{k}) = \int_{\mathbb{R}} \hat{A}(\bar{k},\kappa)\,\hat{\psi}(\kappa)\,d\kappa$. *Right: The frequency kernel* $\hat{A}$ *of a spatially periodic operator* A.

Once again we use perturbation methods to show (i) and (ii). We consider A to have the form $A = A^{\rm o} + \epsilon\, E$, where $A^{\rm o}$ is a spatially invariant operator, E is a spatially periodic operator, and ϵ is a small complex scalar. By using the biinfinite matrix representation, we first find conditions on the spatially invariant operator $A^{\rm o}$ such that (i) and (ii) are satisfied. We then show that (i) and (ii) will *remain* satisfied if ϵ is small enough and if the spatially periodic operator E is "weaker" than $A^{\rm o}$ (in the sense that E is *relatively bounded* with respect to $A^{\rm o}$). The advantage of this approach is that (i) and (ii) are much easier to check for a spatially invariant operator than they are for a spatially periodic one.

3. Frequency representation of periodic operators. In this section we briefly discuss the frequency domain representation of spatially periodic *operators*. We then show how this representation can be used to convert a spatially periodic *system* into a family of matrix-valued LTI systems. For a detailed account the reader is referred to [5, 12].

Let $\hat{\psi}(k)$ and $\hat{\phi}(k)$ denote the Fourier transforms of two spatial functions $\psi(x)$ and $\phi(x)$, respectively. If ψ and ϕ are related by a linear operator A which admits a kernel representation, then $\hat{\psi}$ and $\hat{\phi}$ too are related by a linear operator $\hat{A}$ which admits a kernel representation, and we have

$$\phi(x) \;=\; \int_{\mathbb{R}} A(x,\chi)\,\psi(\chi)\,d\chi \quad \overset{\mathscr{F}_x}{\longleftrightarrow} \quad \hat{\phi}(k) \;=\; \int_{\mathbb{R}} \hat{A}(k,\kappa)\,\hat{\psi}(\kappa)\,d\kappa, \tag{4}$$

where $A(\cdot,\cdot)$ and $\hat{A}(\cdot,\cdot)$ are the *kernel functions* corresponding to the operators A and $\hat{A}$, respectively, and $\mathscr{F}_x$ denotes the Fourier transform. Figure 1 (left) shows a cartoon way of picturing the equation $\hat{\phi} = \hat{A}\,\hat{\psi}$.

A linear operator is called *spatially periodic* with period X if its kernel has the property

$$A(x + \mathrm{X}m, \chi + \mathrm{X}m) \;=\; A(x,\chi) \quad \text{for all} \quad x,\chi \in \mathbb{R},\; m \in \mathbb{Z}.$$

PROPOSITION 1. *Let* A *be a spatially periodic operator with spatial period* $\mathrm{X} = 2\pi/\Omega$. *Then its Fourier transform* $\hat{A}$ *has the kernel representation*

$$\hat{A}(k,\kappa) \;=\; \sum_{l\in\mathbb{Z}} \hat{A}_l(k)\,\delta(k - \kappa - \Omega l). \tag{5}$$

Proof. See [12, Appendix to Chapter 2]. □

Thus the kernel function corresponding to $\hat{A}$ is composed of parallel and equally spaced "impulse sheets" which can be visualized in Figure 1 (right). References [5, 12] further describe how the particular structure (5) of $\hat{A}$ allows the right-hand equation in (4) to be given a biinfinite[1] matrix representation

(6)

$$\begin{bmatrix} \vdots \\ \hat{\phi}(\theta-\Omega) \\ \hat{\phi}(\theta) \\ \hat{\phi}(\theta+\Omega) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \hat{A}_0(\theta-\Omega) & \hat{A}_{-1}(\theta-\Omega) & \hat{A}_{-2}(\theta-\Omega) & \cdots \\ \cdots & \hat{A}_1(\theta) & \hat{A}_0(\theta) & \hat{A}_{-1}(\theta) & \cdots \\ \cdots & \hat{A}_2(\theta+\Omega) & \hat{A}_1(\theta+\Omega) & \hat{A}_0(\theta+\Omega) & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{\psi}(\theta-\Omega) \\ \hat{\psi}(\theta) \\ \hat{\psi}(\theta+\Omega) \\ \vdots \end{bmatrix}$$

for which we adopt the notation

$$\phi_\theta \;=\; \mathcal{A}_\theta\,\psi_\theta, \qquad \theta \in [0,\Omega).$$

In other words, a general spatially periodic operator A can be described by a family of biinfinite matrices $\mathcal{A}_\theta$ parameterized by a variable θ.

Remark 1. We emphasize that the functions $\hat{A}_l(\,\cdot\,)$, $l \in \mathbb{Z}$, in (5) and (6) can be matrix-valued. Thus, in general, $\mathcal{A}_\theta$ has a "block" structure. But throughout this paper and for the sake of brevity we choose not to explicitly refer to this block structure, even though all of our results are derived for matrix-valued $\hat{A}_l(k)$. Similarly, we do not refer explicitly to the Euclidean dimension of the vectors $\hat{\psi}(k)$ and $\hat{\phi}(k)$ for a given $k \in \mathbb{R}$.

Spatially invariant [13] and *spatially periodic pure multiplication* operators constitute special subclasses of spatially periodic operators. In the framework established above, $\mathcal{A}_\theta$ is *diagonal* for spatially invariant operators and *Toeplitz* for periodic pure multiplication operators.

Example 1. The operators $A = \partial_x$ and $F(x) = \cos(\Omega x)$ have the following biinfinite representations

$$\mathcal{A}_\theta \;=\; \operatorname{diag}\{j(\theta+\Omega n)\}_{n\,\in\,\mathbb{Z}}, \qquad \mathcal{F} \;=\; \operatorname{toep}\left\{\ldots,0,\frac{1}{2},\boxed{0},\frac{1}{2},0,\ldots\right\},$$

where the boxed element corresponds to the main diagonal of the biinfinite Toeplitz operator $\mathcal{F}$. Notice that since A is spatially invariant it is fully described by its *Fourier symbol* $\hat{A}_0(k) = jk$, $k \in \mathbb{R}$, and it is the samples of $\hat{A}_0(\cdot)$ at $k = \theta + \Omega n$, $n \in \mathbb{Z}$, that make up the diagonal of $\mathcal{A}_\theta$ for a given θ. We have dropped the θ subscript in $\mathcal{F}$ because it is independent of this variable.

Remark 2. It is possible to define a unitary operator $\mathscr{M}_\theta$, $\theta \in [0,\Omega)$, such that $\psi_\theta = \mathscr{M}_\theta\,\hat{\psi}$, $\phi_\theta = \mathscr{M}_\theta\,\hat{\phi}$, and thus $\mathcal{A}_\theta = \mathscr{M}_\theta\,\hat{A}\,\mathscr{M}_\theta^*$ [12]. $\mathscr{M}_\theta$ is equivalent to a *frequency domain lifting* operation [14, 15, 16] which breaks an $L^2(\mathbb{R})$ function into a family of ℓ^2 vectors. By the unitary property of the lifting operator it follows that

$$(7) \qquad \int_0^\Omega \|\mathcal{A}_\theta\|_{\mathrm{HS}}^2\,d\theta \;=\; \int_0^\Omega \operatorname{trace}[\mathcal{A}_\theta\,\mathcal{A}_\theta^*]\,d\theta \;=\; \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \operatorname{trace}[\hat{A}_l(k)\,\hat{A}_l^*(k)]\,dk,$$

with $\|T\|_{\mathrm{HS}}^2 := \operatorname{trace}[T\,T^*]$ being the square of the Hilbert–Schmidt norm[2] of T.

[1] Sometimes referred to as doubly infinite.

[2] The Hilbert–Schmidt norm of an operator is a generalization of the Frobenius norm of finite-dimensional matrices $\|A\|_{\mathrm{F}}^2 = \sum_{m,n} |a_{mn}|^2 = \operatorname{trace}[AA^*]$.

Finally, given a spatially periodic system in state-space form (1) with spatially periodic operators A, B, and C, one can replace each of these operators with its biinfinite matrix representation to obtain the family of LTI systems (2).

Example 2. Consider the spatially periodic heat equation on the real line

$$\begin{aligned} \partial_t \psi(t,x) &= \left(\partial_x^2 - \alpha \cos(\Omega x)\right) \psi(t,x) + u(t,x), \\ y(t,x) &= \psi(t,x), \end{aligned} \tag{8}$$

with real $\alpha \neq 0$ and $\Omega > 0$.[3] Clearly $A = \partial_x^2 + \alpha \cos(\Omega x)$ with domain

$$\mathscr{D} = \left\{ \phi \in L^2(\mathbb{R}) \mid \phi, \frac{d\phi}{dx} \text{ absolutely continuous}, \frac{d^2\phi}{dx^2} \in L^2(\mathbb{R}) \right\},$$

and B and C are equal to the identity operator on $L^2(\mathbb{R})$. Rewriting the system in its lifted representation, we have

$$\begin{aligned} \partial_t \psi_\theta(t) &= \mathcal{A}_\theta \psi_\theta(t) + u_\theta(t), \\ y_\theta(t) &= \psi_\theta(t), \end{aligned} \tag{9}$$

where $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ are equal to the identity operator on ℓ^2 and $\mathcal{A}_\theta$ can be found from Example 1

$$\mathcal{A}_\theta = - \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & (\theta + \Omega(n-1))^2 & \alpha/2 & 0 & \cdots \\ \cdots & \alpha/2 & (\theta + \Omega n)^2 & \alpha/2 & \cdots \\ \cdots & 0 & \alpha/2 & (\theta + \Omega(n+1))^2 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Notice that (9) is now fully decoupled in the variable θ. In other words, (8) is equivalent to the family of state-space representations (9) parameterized by $\theta \in [0, \Omega)$.

4. Problem setup. In this section we set the stage for the analysis that will follow in the proceeding sections. We first describe the class of spatially periodic LTI systems that we will be considering in this paper. We then review some facts and definitions regarding the spectrum of the infinitesimal generators of such systems, their exponential stability, and conditions under which the spectrum determines exponential stability. Finally, we introduce the notion of the $\mathcal{H}^2$ norm for exponentially stable spatially periodic systems and give a procedure for computing this norm.

Let us consider a distributed system of the form

$$\begin{aligned} \partial_t \psi(t,x) &= A\, \psi(t,x) + B\, u(t,x) \\ &= \left(A^{\mathrm{o}} + B^{\mathrm{o}} \epsilon F C^{\mathrm{o}}\right) \psi(t,x) + B\, u(t,x), \\ y(t,x) &= C\, \psi(t,x), \end{aligned} \tag{10}$$

where $t \in [0, \infty)$ and $x \in \mathbb{R}$ with the following assumptions. The (possibly unbounded) operators A^{o}, B^{o}, and C^{o} are spatially invariant, and the bounded operators B and C are spatially periodic with period $\mathrm{X} = 2\pi/\Omega$. $F(x) = 2L\cos(\Omega x)$, with L a constant

[3] By $\partial_t\psi(t,x)$ and $\partial_x^2\psi(t,x)$ we mean the spatiotemporal functions $\partial_t\psi$ and $\partial_x^2\psi$, respectively, evaluated at the point (t,x).

matrix, and ϵ is a complex scalar. $A^{\rm o}$, $B^{\rm o}$, $C^{\rm o}$, and $E := B^{\rm o} F C^{\rm o}$ are all defined on a common dense domain $\mathscr{D} \subset L^2(\mathbb{R})$. $A = A^{\rm o} + \epsilon E$ is closed and generates a *strongly continuous* semigroup (also known as a C_0 semigroup) e^{At} [2]. We will refer to A as the *infinitesimal generator* of the system. The functions u, y, and ψ are, respectively, the spatiotemporal input, output, and state of the system and belong to $L^2(\mathbb{R})$ for all t.

Comment on notation. To avoid clutter, we henceforth drop the "ˆ" from the representation of frequency domain functions. For example, we use $A_0(\,\cdot\,)$ [instead of $\hat{A}_0(\,\cdot\,)$] to represent the Fourier symbol of the spatially invariant operator $A^{\rm o}$.

As shown in [12, 5, 17], system (10) can be represented in the (spatial) frequency domain by the family of systems

$$\begin{aligned} \partial_t\, \psi_\theta(t) \; &= \left(\mathcal{A}_\theta^{\rm o} \,+\, \epsilon\, \mathcal{B}_\theta^{\rm o}\, \mathcal{F}\, \mathcal{C}_\theta^{\rm o}\right) \psi_\theta(t) \,+\, \mathcal{B}_\theta\, u_\theta(t) \\ &= \left(\mathcal{A}_\theta^{\rm o} \,+\, \epsilon\, \mathcal{E}_\theta\right) \psi_\theta(t) \,+\, \mathcal{B}_\theta\, u_\theta(t), \\ y_\theta(t) \; &= \mathcal{C}_\theta\, \psi_\theta(t), \end{aligned} \tag{11}$$

parameterized by $\theta \in [0, \Omega)$. The vectors u_θ, y_θ, and ψ_θ belong to ℓ^2 for all t. $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ have no particular structure and can be any bounded operators on ℓ^2. $\mathcal{F}$ has the form given in Example 1 with $\frac{1}{2}$ replaced by the matrix L. $\mathcal{A}_\theta^{\rm o}$, $\mathcal{B}_\theta^{\rm o}$, and $\mathcal{C}_\theta^{\rm o}$ are (possibly unbounded) block diagonal operators on ℓ^2,

$$\mathcal{A}_\theta^{\rm o} \;=\; \begin{bmatrix} \ddots & & \\ & A_0(\theta_n) & \\ & & \ddots \end{bmatrix}, \qquad \mathcal{B}_\theta^{\rm o} \;=\; \begin{bmatrix} \ddots & & \\ & B^{\rm o}(\theta_n) & \\ & & \ddots \end{bmatrix}, \qquad \mathcal{C}_\theta^{\rm o} \;=\; \begin{bmatrix} \ddots & & \\ & C^{\rm o}(\theta_n) & \\ & & \ddots \end{bmatrix},$$

$$\mathcal{E}_\theta \;:=\; \mathcal{B}_\theta^{\rm o}\, \mathcal{F}\, \mathcal{C}_\theta^{\rm o} \;=\; \begin{bmatrix} \ddots & \ddots & & \\ \ddots & 0 & A_{-1}(\theta_n) & \\ & A_1(\theta_{n+1}) & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \tag{12}$$

with $\theta_n := \theta + \Omega n$, $n \in \mathbb{Z}$, and[4]

$$A_1(\,\cdot\,) \;:=\; B^{\rm o}(\,\cdot\,)\, L\, C^{\rm o}(\,\cdot\, - \Omega), \qquad A_{-1}(\,\cdot\,) \;:=\; B^{\rm o}(\,\cdot\,)\, L\, C^{\rm o}(\,\cdot\, + \Omega). \tag{13}$$

$A_0(\,\cdot\,)$, $B^{\rm o}(\,\cdot\,)$, and $C^{\rm o}(\,\cdot\,)$ denote the Fourier symbols of the spatially invariant operators $A^{\rm o}$, $B^{\rm o}$, and $C^{\rm o}$, respectively.

In sections 7 and 5 we will establish conditions for exponential stability of system (10) and compute its $\mathcal{H}^2$ norm. We will see that the biinfinite representation (11)–(12) will play a key role in simplifying the perturbation analysis of system (10).

Remark 3. Taking $F(x)$ to be a pure cosine is not restrictive. The results obtained here can be easily extended to problems where $F(x)$ is any periodic function with absolutely convergent Fourier series coefficients.

Remark 4. The system (10) can be considered as the linear fractional transformation [18] of a spatially periodic system $G^{\rm o}$ with *spatially invariant* infinitesimal

[4] We emphasize that the notational convention used in the elements of $\mathcal{E}_\theta$ is the same as that used in (6); the nth row of $\mathcal{E}_\theta$ is $\{\ldots, 0, A_1(\theta_n), 0, A_{-1}(\theta_n), 0, \ldots\}$.

generator $A^{\rm o}$,

$$G^{\rm o} = \left[\begin{array}{c|cc} A^{\rm o} & B & B^{\rm o} \\ \hline C & 0 & 0 \\ C^{\rm o} & 0 & 0 \end{array}\right],$$

and the (memoryless and bounded) spatially periodic pure multiplication operator $\epsilon F(x) = \epsilon\, 2L\cos(\Omega x)$.

Sectorial operators and exponential stability. We introduce the class of *sectorial* operators. These operators generate *holomorphic* (also known as *analytic*) semigroups e^{At}. Sectorial operators have the important property that their spectrum determines the decay or growth rate of their semigroup; i.e., they satisfy the spectrum-determined growth condition.

Suppose A is densely defined, $\rho(A)$ contains a sector of the complex plane $|\arg(z-\alpha)| \leq \frac{\pi}{2} + \gamma$, $\gamma > 0$, $\alpha \in \mathbb{R}$, and there exists some $M > 0$ such that

$$\|(zI - A)^{-1}\| \;\leq\; \frac{M}{|z-\alpha|} \quad \text{for} \quad |\arg(z-\alpha)| \leq \frac{\pi}{2} + \gamma. \tag{14}$$

Then A generates a holomorphic semigroup, and we write $A \in \mathscr{H}(\gamma, \alpha, M)$ [11, 9]. We say that A is *sectorial* if A belongs to some $\mathscr{H}(\gamma, \alpha, M)$.

A semigroup is called exponentially stable if there exist positive constants M and ϱ such that [2]

$$\|e^{At}\| \;\leq\; M\, e^{-\varrho t} \quad \text{for} \quad t \geq 0.$$

The following theorem constitutes the reason for our interest in sectorial operators and forms the foundation of our analysis in section 7.

THEOREM 2. *Assume that A is sectorial. Then A generates an exponentially stable C_0 semigroup if and only if $\Sigma(A) \subset \mathbb{C}^-$.*

Proof. If A is sectorial, it defines a holomorphic C_0 semigroup and e^{At} is differentiable for $t > 0$ [10, 19]. Then [8] shows that this is sufficient for the spectrum-determined growth condition to hold. Since $\Sigma(A) \subset \mathbb{C}^-$ and $\Sigma(A)$ belongs to a left sector, it follows that $\Sigma(A)$ is bounded away from the imaginary axis. Let $\omega_\sigma = \sup_{z\in\Sigma(A)} \mathrm{Re}(z)$. Then $\omega_\sigma < 0$, and A generates an exponentially stable C_0 semigroup. Clearly $\Sigma(A) \subset \mathbb{C}^-$ is also a necessary condition for exponentially stability, and the proof is complete. ∎

Spectrum of spatially periodic operators. We show how the spectrum of A relates to the spectrum of the θ-parameterized family of operators $\mathcal{A}_\theta$. We will use this in section 7 to find the spectrum of A, as needed in Theorem 2 to establish exponential stability.

Since the spatially periodic operator A and the family of biinfinite matrices $\mathcal{A}_\theta$, $\theta \in [0, \Omega)$, are related via a unitary transformation, it follows that [5]

$$\Sigma(A) \;=\; \overline{\bigcup_{\theta\in[0,\Omega)} \Sigma(\mathcal{A}_\theta)}. \tag{15}$$

In the case where A is spatially invariant (and thus $\mathcal{A}_\theta = \mathrm{diag}\{\ldots, A_0(\theta_n), \ldots\}$), (15) further simplifies to

$$\Sigma(A) \;=\; \overline{\bigcup_{k\in\mathbb{R}} \Sigma_{\rm p}\big(A_0(k)\big)}. \tag{16}$$

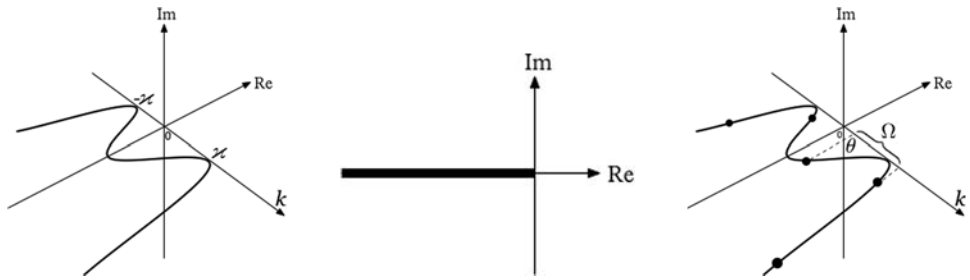

FIG. 2. *Left: The symbol $A_0(\cdot)$ of Example 3 in "complex-plane × spatial-frequency" space. Center: The spectrum $\Sigma(A)$ in the complex plane. Right: For spatially invariant A, the (diagonal) elements of $\mathcal{A}_\theta$ are samples of the Fourier symbol $A_0(\cdot)$.*

Example 3. Let $A = -(\partial_x^2 + \varkappa^2)^2$ with domain

$$\mathscr{D} = \left\{ \phi \in L^2(\mathbb{R}) \mid \phi, \frac{d\phi}{dx}, \frac{d^2\phi}{dx^2}, \frac{d^3\phi}{dx^3} \text{ absolutely continuous}, \frac{d^4\phi}{dx^4} \in L^2(\mathbb{R}) \right\}. \tag{17}$$

Integration by parts shows that A is a self-adjoint operator and thus closed. The function $A_0(k) = -(k^2 - \varkappa^2)^2$ is the Fourier symbol of A; see Figure 2 (left). Since $A_0(\cdot)$ is scalar, we have $\Sigma_\mathrm{p}\big(A_0(k)\big) = A_0(k)$ for every k. It is easy to see that $A_0(\cdot)$ takes every real negative value, and thus from (16) A has continuous spectrum $\Sigma(A) = (-\infty, 0\,]$; see Figure 2 (center).

Remark 5. When A is spatially invariant, a helpful way to think about $\Sigma(A)$ in terms of the symbol $A_0(\cdot)$ of A is suggested by the previous example. First plot $\Sigma_\mathrm{p}\big(A_0(\cdot)\big)$ in the "complex-plane × spatial-frequency" space, as in Figure 2 (left). Then the orthogonal projection onto the complex plane of this plot would give $\Sigma(A)$. This can be considered as a geometric interpretation of (16). In Example 3, since $A_0(\cdot)$ is real scalar and takes all negative values, this projection yields the negative real axis. But, in general, if $A_0(\cdot) \in \mathbb{C}^{q\times q}$, this projection would lead to q curves in the complex plane. Notice also that in this setting $\Sigma(\mathcal{A}_\theta)$ is the projection onto the complex plane of samples of $\Sigma_\mathrm{p}\big(A_0(\cdot)\big)$ taken at $k = \theta + \Omega n$, $n \in \mathbb{Z}$, in the complex-plane × spatial-frequency space. As θ varies in $[0, \Omega)$ these projections trace out $\Sigma(A)$ in the complex plane. This can be considered as a geometric interpretation of (15). Figure 2 (right) shows the said samples in the complex-plane × spatial-frequency space for a scalar A.

$\mathcal{H}^2$ norm of spatially periodic systems. We define the $\mathcal{H}^2$ norm of an exponentially stable spatially periodic system G as

$$\|G\|_{\mathcal{H}^2}^2 \;=\; \frac{1}{2\pi}\int_0^\Omega \int_0^\infty \operatorname{trace}[\mathcal{G}_\theta(t)\,\mathcal{G}_\theta^*(t)]\,dt\,d\theta,$$

where $\mathcal{G}_\theta(t) = \mathcal{C}_\theta\, e^{\mathcal{A}_\theta t}\,\mathcal{B}_\theta$ is the impulse response of the system (11). The proof of the following theorem can be found in [5, 12].

THEOREM 3. *Consider the exponentially stable spatially periodic LTI system G, with spatial period $\mathrm{X} = 2\pi/\Omega$ and state-space realization* (10)–(11)*. Then*

$$\|G\|_{\mathcal{H}^2}^2 \;=\; \frac{1}{2\pi}\int_0^\Omega \operatorname{trace}[\mathcal{C}_\theta \mathcal{P}_\theta\, \mathcal{C}_\theta^*]\,d\theta \;=\; \frac{1}{2\pi}\int_0^\Omega \operatorname{trace}[\mathcal{B}_\theta^*\, \mathcal{Q}_\theta\, \mathcal{B}_\theta]\,d\theta,$$

where $\mathcal{P}_\theta$ and $\mathcal{Q}_\theta$ are the solutions of the θ-parameterized algebraic Lyapunov equations

$$\mathcal{A}_\theta \mathcal{P}_\theta + \mathcal{P}_\theta \mathcal{A}_\theta^* = -\mathcal{B}_\theta \mathcal{B}_\theta^*, \qquad \mathcal{A}_\theta^* \mathcal{Q}_\theta + \mathcal{Q}_\theta \mathcal{A}_\theta = -\mathcal{C}_\theta^* \mathcal{C}_\theta.$$

5. Perturbation analysis of the $\mathcal{H}^2$ norm. The difficulty in calculating the $\mathcal{H}^2$ norm using Theorem 3 is that, unless $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{C}_\theta$ are diagonal, the operators $\mathcal{P}_\theta$ and $\mathcal{Q}_\theta$ are "full," meaning that they possess *all* of their (infinite number of) subdiagonals. This makes the computation of the $\mathcal{H}^2$ norm numerically expensive. Namely, one has to solve an *infinite-dimensional* algebraic Lyapunov equation to find the operator $\mathcal{P}_\theta$ (or $\mathcal{Q}_\theta$) *for every value of* $\theta \in [0, \Omega)$. In this section we will see how one can use analytic perturbation techniques to compute the $\mathcal{H}^2$ norm in a simple and numerically efficient way and without having to explicitly find the full $\mathcal{P}_\theta$ and $\mathcal{Q}_\theta$ operators. Such a perturbation analysis is very useful in predicting general trends and extracting valuable information about the $\mathcal{H}^2$ norm, as needed in the design of periodic feedback.

Consider the general setup of (10), where we take ϵ to be a small *real* scalar. We assume that $A = A^{\mathrm{o}} + \epsilon E$ defines an exponentially stable C_0 semigroup on $L^2(\mathbb{R})$ with finite $\mathcal{H}^2$ norm for small enough ϵ and that B and C are spatially invariant operators. We are interested in the changes in the $\mathcal{H}^2$ norm of this system for small magnitudes of ϵ and different values of the frequency Ω.

Let

$$\mathcal{P}_\theta(\epsilon) = \mathcal{P}_\theta^{(0)} + \epsilon \mathcal{P}_\theta^{(1)} + \epsilon^2 \mathcal{P}_\theta^{(2)} + \cdots,$$

with $\mathcal{P}_\theta^*(\epsilon) = \mathcal{P}_\theta(\epsilon)$. This implies that $\mathcal{P}_\theta^{(m)*} = \mathcal{P}_\theta^{(m)}$ for all $m = 0, 1, 2, \ldots$; i.e., $\mathcal{P}_\theta^{(m)}$ are self-adjoint operators for all m. The proof of convergence of the above series is relegated to the appendix. Our aim is to find $\mathcal{P}_\theta^{(m)}$ by solving the Lyapunov equation

$$\mathcal{A}_\theta(\epsilon) \mathcal{P}_\theta(\epsilon) + \mathcal{P}_\theta(\epsilon) \mathcal{A}_\theta^*(\epsilon) = -\mathcal{B}_\theta \mathcal{B}_\theta^* \tag{18}$$

$$\Downarrow$$

$$\begin{aligned}(\mathcal{A}_\theta^{\mathrm{o}} + \epsilon \mathcal{E}_\theta)(\mathcal{P}_\theta^{(0)} + \epsilon \mathcal{P}_\theta^{(1)} + \epsilon^2 \mathcal{P}_\theta^{(2)} + \cdots)& \\ + (\mathcal{P}_\theta^{(0)} + \epsilon \mathcal{P}_\theta^{(1)} + \epsilon^2 \mathcal{P}_\theta^{(2)} + \cdots)(\mathcal{A}_\theta^{\mathrm{o}} + \epsilon \mathcal{E}_\theta)^* &= -\mathcal{B}_\theta \mathcal{B}_\theta^*\end{aligned} \tag{19}$$

and to compute the $\mathcal{H}^2$ norm of the system by using Theorem 3 and

$$\|G\|_{\mathcal{H}^2}^2 = \frac{1}{2\pi} \int_0^\Omega \operatorname{trace}[\mathcal{C}_\theta \mathcal{P}_\theta(\epsilon) \mathcal{C}_\theta^*] \, d\theta.$$

By equating equal powers of ϵ on both sides of (19), we have

$$\mathcal{A}_\theta^{\mathrm{o}} \mathcal{P}_\theta^{(0)} + \mathcal{P}_\theta^{(0)} \mathcal{A}_\theta^{\mathrm{o}*} = -\mathcal{B}_\theta \mathcal{B}_\theta^*, \tag{20}$$

$$\mathcal{A}_\theta^{\mathrm{o}} \mathcal{P}_\theta^{(1)} + \mathcal{P}_\theta^{(1)} \mathcal{A}_\theta^{\mathrm{o}*} = -\left(\mathcal{E}_\theta \mathcal{P}_\theta^{(0)} + \mathcal{P}_\theta^{(0)} \mathcal{E}_\theta^*\right), \tag{21}$$

$$\mathcal{A}_\theta^{\mathrm{o}} \mathcal{P}_\theta^{(2)} + \mathcal{P}_\theta^{(2)} \mathcal{A}_\theta^{\mathrm{o}*} = -\left(\mathcal{E}_\theta \mathcal{P}_\theta^{(1)} + \mathcal{P}_\theta^{(1)} \mathcal{E}_\theta^*\right), \tag{22}$$

$$\vdots$$

The existence of a unique solution to each of these equations is guaranteed by the exponential stability of the unperturbed system. Furthermore, in (20) since the operators $\mathcal{A}_\theta^{\mathrm{o}}$ and $\mathcal{B}_\theta \mathcal{B}_\theta^*$ are diagonal, so is $\mathcal{P}_\theta^{(0)}$. In (21) the right-hand side operator has the structure of being nonzero only on the first upper and lower subdiagonals, and hence $\mathcal{P}_\theta^{(1)}$ inherits the same structure since $\mathcal{A}_\theta^\circ$ is diagonal. In the same manner, the operator $\mathcal{P}_\theta^{(2)}$ is nonzero only on the main diagonal and on the second upper and lower subdiagonals. This type of argument can be applied to all other $\mathcal{P}_\theta^{(m)}$, $m = 3, 4, \ldots,$ and we have

$$\mathcal{P}_\theta^{(0)} = \begin{bmatrix} \ddots & & \\ & P_0(\theta_n) & \\ & & \ddots \end{bmatrix}, \qquad \mathcal{P}_\theta^{(1)} = \begin{bmatrix} \ddots & \ddots & & \\ \ddots & 0 & P_1^*(\theta_n) & \\ & P_1(\theta_n) & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

$$\mathcal{P}_\theta^{(2)} = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \ddots & 0 & P_2^*(\theta_{n+1}) & \\ \ddots & 0 & Q_0(\theta_n) & 0 & \ddots \\ & P_2(\theta_{n+1}) & 0 & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \qquad \ldots,$$

where we have used the self-adjointness of these operators in writing the elements of their upper subdiagonals.

Remark 6. Note that not only is $\mathcal{P}_\theta^{(m)}$ *not* a full operator, it has at most m nonzero upper and lower subdiagonals. Furthermore, all $\mathcal{P}_\theta^{(m)}$ for odd m have zero diagonal and are thus trace-free operators.

Remark 7. Although $\mathcal{A}_\theta = \mathcal{A}_\theta^\circ + \epsilon\,\mathcal{E}_\theta$ has only one nonzero subdiagonal, the operator $\mathcal{P}_\theta(\epsilon) = \mathcal{P}_\theta^{(0)} + \epsilon\,\mathcal{P}_\theta^{(1)} + \cdots$ possesses all of its subdiagonals. This is precisely the reason why direct calculation of $\mathcal{P}_\theta$ in Theorem 3 is computationally difficult.

From (20)–(22) the operators $\mathcal{P}_\theta^{(0)}$, $\mathcal{P}_\theta^{(1)}$, and $\mathcal{P}_\theta^{(2)}$ are found by equating, element by element, the biinfinite matrices on both sides of these equations. For example, (20) leads to

$$A_0(\theta+\Omega n)\,P_0(\theta+\Omega n) \;+\; P_0(\theta+\Omega n)\,A_0^*(\theta+\Omega n) \;=\; -\,B(\theta+\Omega n)\,B^*(\theta+\Omega n)$$

for every $n \in \mathbb{Z}$ and $\theta \in [0, \Omega)$. But as n takes all integer values and θ changes in $[0, \Omega)$, the variable $k = \theta + \Omega n$ takes all real values, and one can rewrite the above equation as

$$A_0(k)\,P_0(k) \;+\; P_0(k)\,A_0^*(k) \;=\; -\,B(k)\,B^*(k),$$

with $k \in \mathbb{R}$. By applying the same procedure to (21)–(22) one arrives at

$$A_0(k)\,P_0(k) \;+\; P_0(k)\,A_0^*(k) \;=\; -B(k)\,B^*(k), \tag{23}$$

$$A_0(k)\,P_1(k) \;+\; P_1(k)\,A_0^*(k-\Omega) \;=\; -\big(A_1(k)\,P_0(k-\Omega) \;+\; P_0(k)\,A_{-1}^*(k-\Omega)\big), \tag{24}$$

$$\begin{aligned} A_0(k)\,Q_0(k) \;+\; Q_0(k)\,A_0^*(k) \;=\; -\big(&A_1(k)\,P_1^*(k) \;+\; P_1(k)\,A_1^*(k) \\ &+\; A_{-1}(k)\,P_1(k+\Omega) \;+\; P_1^*(k+\Omega)\,A_{-1}^*(k)\big), \end{aligned} \tag{25}$$

and so on for all nonzero diagonals of $\mathcal{P}_\theta^{(m)}$, $m = 3, 4, \ldots.$ The existence of a unique solution to each of these equations is guaranteed by the exponential stability of the unperturbed system.

From the above equations one first finds $P_0(\,\cdot\,)$ from (23), then $P_1(\,\cdot\,)$ from (24), and so on. In other words, computing the subdiagonals of $\mathcal{P}_\theta$ becomes *decoupled in one direction*. This decoupling would not have been possible had we not employed a perturbation approach and had attempted to solve (18) directly.

Returning to the calculation of the $\mathcal{H}^2$ norm, let us first separate the diagonal part of $\mathcal{P}_\theta^{(2)}$ by rewriting it as $\mathcal{P}_\theta^{(2)} = \overline{\mathcal{P}}_\theta^{(2)} + \widetilde{\mathcal{P}}_\theta^{(2)}$, where

$$\overline{\mathcal{P}}_\theta^{(2)} := \begin{bmatrix} \ddots & & \\ & Q_0(\theta_n) & \\ & & \ddots \end{bmatrix}$$

and $\widetilde{\mathcal{P}}_\theta^{(2)}$ contains the rest of $\mathcal{P}_\theta^{(2)}$. Clearly $\mathrm{trace}[\mathcal{C}_\theta\, \widetilde{\mathcal{P}}_\theta^{(2)}\, \mathcal{C}_\theta^*] = 0$. Also recall that

$$\mathrm{trace}[\mathcal{C}_\theta\, \mathcal{P}_\theta^{(2m+1)}\, \mathcal{C}_\theta^*] = 0, \qquad m = 0, 1, 2, \ldots. \tag{26}$$

Therefore

$$\begin{aligned} \|G\|_{\mathcal{H}^2}^2 &= \frac{1}{2\pi}\int_0^\Omega \mathrm{trace}[\mathcal{C}_\theta\, \mathcal{P}_\theta(\epsilon)\, \mathcal{C}_\theta^*]\, d\theta \\ &= \frac{1}{2\pi}\int_0^\Omega \mathrm{trace}[\mathcal{C}_\theta\, \mathcal{P}_\theta^{(0)}\, \mathcal{C}_\theta^* \,+\, \epsilon^2\, \mathcal{C}_\theta\, \mathcal{P}_\theta^{(2)}\, \mathcal{C}_\theta^*]\, d\theta \,+\, O(\epsilon^4) \\ &= \frac{1}{2\pi}\int_0^\Omega \mathrm{trace}[\mathcal{C}_\theta\, \mathcal{P}_\theta^{(0)}\, \mathcal{C}_\theta^* \,+\, \epsilon^2\, \mathcal{C}_\theta\, \overline{\mathcal{P}}_\theta^{(2)}\, \mathcal{C}_\theta^*]\, d\theta \,+\, O(\epsilon^4), \end{aligned}$$

where the absence of odd powers of ϵ is due to (26) and the last equation follows from the fact that $\mathrm{trace}[\mathcal{C}_\theta\, \widetilde{\mathcal{P}}_\theta^{(2)}\, \mathcal{C}_\theta^*] = 0$. By using the unitary property of the lifting transform we have

$$\begin{aligned} \int_0^\Omega \mathrm{trace}[\mathcal{C}_\theta\, \mathcal{P}_\theta^{(0)}\, \mathcal{C}_\theta^*]\, d\theta &= \int_{-\infty}^\infty \mathrm{trace}[C(k)\, P_0(k)\, C^*(k)]\, dk, \\ \int_0^\Omega \mathrm{trace}[\mathcal{C}_\theta\, \overline{\mathcal{P}}_\theta^{(2)}\, \mathcal{C}_\theta^*]\, d\theta &= \int_{-\infty}^\infty \mathrm{trace}[C(k)\, Q_0(k)\, C^*(k)]\, dk, \end{aligned}$$

and we arrive at the final result

$$\|G\|_{\mathcal{H}^2}^2 = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{trace}[C(k)P_0(k)C^*(k) \,+\, \epsilon^2\, C(k)Q_0(k)C^*(k)]\, dk \,+\, O(\epsilon^4). \tag{27}$$

We have thus proved the following theorem, which is the main result of this section.

THEOREM 4. *Consider the exponentially stable spatially periodic LTI system G with finite $\mathcal{H}^2$ norm, spatial period* $\mathrm{X} = 2\pi/\Omega$, *and state-space realization* (10). *Then for small values of $|\epsilon|$ the $\mathcal{H}^2$ norm of the system* (10) *can be computed from* (27), *where $P_0(\,\cdot\,)$ and $Q_0(\,\cdot\,)$ are solutions of the family of finite-dimensional Lyapunov and Sylvester equations described by* (23)–(25).

The described procedure can be continued to find higher-order terms in the perturbation series of the $\mathcal{H}^2$ norm. The interested can refer to [20] for details.

6. Examples. As an application of the perturbation results of the previous section, we first investigate the occurrence of "parametric resonance" for a class of spatially periodic systems. Parametric resonance occurs when a specific frequency Ω_{res}

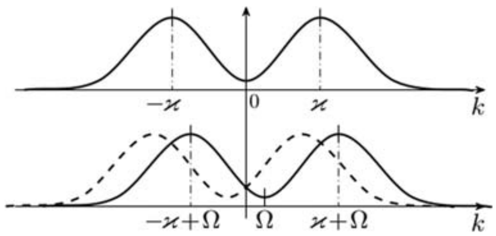

FIG. 3. *Top: Plot of* $P_0(\,\cdot\,)$*. Bottom: Plot of* $P_0(\,\cdot\,-\Omega)$ *and* $P_0(\,\cdot\,+\Omega)$ *(dashed line).*

of the periodic perturbation resonates with some "natural frequency" $\varkappa$ of the unperturbed system, leading to a local (in Ω) change in system behavior [3]. In the systems we consider in this section this change in behavior is captured by the value of the $\mathcal{H}^2$ norm.

Example 4. We consider the Swift–Hohenberg equation, which is of interest in hydrodynamics [21, 22, 23] and nonlinear optics [24, 25] as well as in other branches of physics [23]. The linearization of the Swift–Hohenberg equation around its time-independent spatially periodic solution leads to a PDE with spatially periodic coefficients of the form [26]

$$\begin{aligned} \partial_t \psi &= -(\partial_x^2 + \varkappa^2)^2\,\psi - c\,\psi + f\,\psi + u, \\ y &= \psi, \end{aligned} \tag{28}$$

with $0 \neq \varkappa \in \mathbb{R}$, $c > 0$, and $f(x) = f(x + 2\pi/\Omega)$. We assume here that $f(x) = \epsilon\cos(\Omega x)$, with $\epsilon \in \mathbb{R}$ small. By comparing (28) and (10) we have

$$A_0(k) = -(k^2 - \varkappa^2)^2 - c, \quad B^{\mathrm{o}}(k) = 1, \quad C^{\mathrm{o}}(k) = 1, \quad B(k) = 1, \quad C(k) = 1, \quad L = \frac{1}{2}.$$

For this system the functions $P_0(k)$ and $Q_0(k)$ of the previous section simplify to[5]

$$P_0(k) = \frac{-1}{2A_0(k)}, \tag{29}$$

$$\begin{aligned} Q_0(k) &= \frac{1}{\big(A_0(k)\big)^2}\left(\frac{-1}{2A_0(k-\Omega)} + \frac{-1}{2A_0(k+\Omega)}\right) \\ &= 4\big(P_0(k)\big)^2\,\big(P_0(k-\Omega) + P_0(k+\Omega)\big), \end{aligned} \tag{30}$$

and our aim is to find the $\mathcal{H}^2$ norm

$$\|G\|^2_{\mathcal{H}^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\big(P_0(k) + \epsilon^2\,Q_0(k)\big)\,dk + O(\epsilon^4) \tag{31}$$

for different values of the parameter $\Omega > 0$. More specifically, we are interested in the values of Ω for which the $\mathcal{H}^2$ norm is maximized.

From (29) we have $P_0(k) = (1/2)/[(k^2 - \varkappa^2)^2 + c]$. The first plot of Figure 3 shows $P_0(\,\cdot\,)$, while the second shows $P_0(\,\cdot\,-\Omega)$ and $P_0(\,\cdot\,+\Omega)$ [dashed line] for a given value of $\Omega \neq 0$. As Ω is increased, $P_0(\,\cdot\,-\Omega)$ slides to the right and $P_0(\,\cdot\,+\Omega)$ to the left. From (30) it is clear that, to find $Q_0(\,\cdot\,)$ for a given Ω, one would sum the two functions in the second plot and multiply the result by the square of the first

[5]To find $Q_0(k)$ one needs to first find $P_1(k)$, but we have omitted the details for brevity.

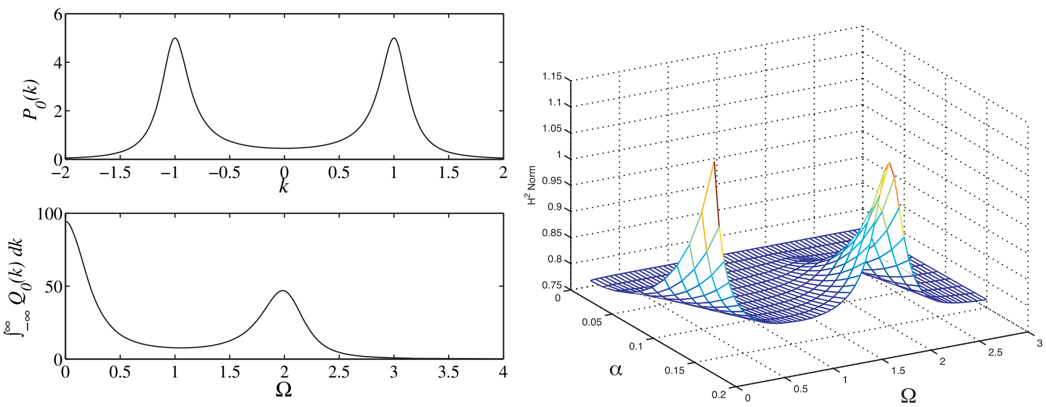

FIG. 4. *Left: Plots of Example 4 for $\varkappa = 1$ and $c = 0.1$. Notice that the first graph is plotted against k and the second against Ω. Right: The plot of the $\mathcal{H}^2$ norm of the same example but calculated by taking large truncations of the $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{C}_\theta$ matrices and using Theorem* 3.

plot. The interesting question now is for what value(s) of $\Omega \in (0, \infty)$ the $\mathcal{H}^2$ norm in (31) would be maximized.

One can easily see that as $\Omega \to 0$ the peaks of $P_0(\cdot - \Omega)$ and $P_0(\cdot + \Omega)$ merge toward those of $(P_0(\cdot))^2$. Thus $\int_{-\infty}^{\infty} Q_0(k)\, dk$ grows, and hence $\|G\|_{\mathcal{H}^2}^2$ grows.[6] This is not surprizing; as $\Omega \to 0$ the perturbation is tending toward a constant function $F(x) = \cos(\Omega x) \to 1$. This results in shifting the whole spectrum of A^o toward the right half of the complex plane, thus increasing the $\mathcal{H}^2$ norm of the perturbed system.

But we are more interested in frequencies $\Omega \gg 0$ that lead to a local (in Ω) increase in the $\mathcal{H}^2$ norm. Notice that a different alignment of the peaks can also occur, which leads to another local maximum of the $\mathcal{H}^2$ norm as a function of Ω. This happens when the peak of $P_0(\cdot - \Omega)$ at $k = -\varkappa + \Omega$ becomes aligned with the peak of $(P_0(\cdot))^2$ at $k = \varkappa$ and, simultaneously, the peak of $P_0(\cdot + \Omega)$ at $k = \varkappa - \Omega$ becomes aligned with the peak of $(P_0(\cdot))^2$ at $k = -\varkappa$. Clearly this occurs when

$$-\varkappa + \Omega_{\text{res}} \;=\; \varkappa \quad \Longrightarrow \quad \Omega_{\text{res}} \;=\; 2\,\varkappa.$$

This result agrees exactly with that obtained in [27], where in the analysis of the same problem it is shown that the part of the spectrum of A closest to the imaginary axis "resonates" with perturbations whose frequency satisfies the relation $\Omega = 2\,\varkappa$.

Consider (28) with $\varkappa = 1$ and $c = 0.1$. For this system $\int_{-\infty}^{\infty} P_0(k)\, dk \approx 4.74$. Figure 4 (left) shows the graphs of $P_0(\cdot)$ plotted against k and $\int_{-\infty}^{\infty} Q_0(k)\, dk$ plotted against Ω. The peak at $\Omega = 2$ in the lower plot verifies the relation $\Omega_{\text{res}} = 2\,\varkappa$ obtained previously.

Figure 4 (right) shows the $\mathcal{H}^2$ norm of this system computed by taking large enough truncations [5] of the $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{C}_\theta$ matrices and then applying Theorem 3. The figure shows that the trends were indeed correctly predicted by the perturbation analysis and the peaks at $\Omega = 0, 2$ correspond to the peaks of $\int_{-\infty}^{\infty} Q_0(k)\, dk$.

Now consider (28) with $\varkappa = 1$ and $c = 0.1$ but with ϵ replaced by ϵj. This would correspond to $L = j/2$ in the notation of (10). Obviously the unperturbed system

[6] Remember that $P_0(k)$ is independent of Ω, and thus $\int_{-\infty}^{\infty} P_0(k)\, dk$ remains constant for different Ω.

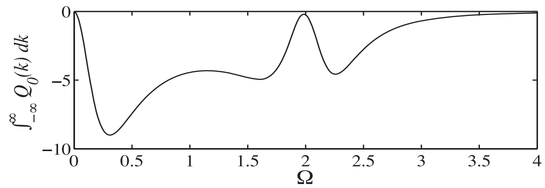

FIG. 5. *The plot of Example* 4 *for* $\varkappa = 1$, $c = 0.1$, *and a purely imaginary perturbation.*

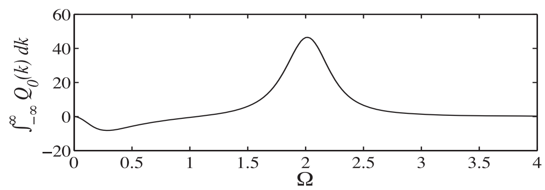

FIG. 6. *Plot of Example* 5 *for* $\varkappa = 1$ *and* $c = 0.1$.

remains the same as before, and hence $\int_{-\infty}^{\infty} P_0(k)\,dk \approx 4.74$. Figure 5 shows the graph of $\int_{-\infty}^{\infty} Q_0(k)\,dk$ which demonstrates that for this system the purely imaginary perturbation reduces the $\mathcal{H}^2$ norm at all frequencies. The physical interpretation of such a perturbation is investigated in [17].

Example 5. We consider a slightly different version of the Swift–Hohenberg equation in the previous example [5]

$$\begin{aligned} \partial_t \psi &= -(\partial_x^2 + \varkappa^2)^2 \psi - c\psi + \epsilon \cos(\Omega x)\,\partial_x \psi + u, \\ y &= \psi. \end{aligned} \tag{32}$$

By comparing (32) and (10) we have

$$A_0(k) = -(k^2 - \varkappa^2)^2 - c, \quad B^{\mathrm{o}}(k) = 1, \quad C^{\mathrm{o}}(k) = jk, \quad B(k) = 1, \quad C(k) = 1, \quad L = \frac{1}{2}. \tag{33}$$

The difference between this system and (28) is that here C^{o} is a spatial derivative. The plot of Figure 6 demonstrates $\int_{-\infty}^{\infty} Q_0(k)\,dk$ for $\varkappa = 1$ and $c = 0.1$. Notice that the peak at $\Omega_{\mathrm{res}} = 2$ remains the same as in Figure 4, but we now have a decrease at small frequencies. This is due to the derivative operator $C^{\mathrm{o}} = \partial_x$. These results are in agreement with the (nonperturbation) calculations for the same system in Example B, section VII of [5]. Our perturbation methods correctly predict the increase at $\Omega = 2$ and the decrease around $\Omega \approx 0.4$ of the $\mathcal{H}^2$ norm.

Example 6. The system in this example is inspired by boundary layer and channel flow problems, where the introduction of corrugated walls or periodic body forces influences drag reduction or enhancement in such geometries. The following PDE has an analogous structure to the linearized Navier–Stokes equations in these geometries. Consider

$$A_0(k) = \begin{bmatrix} -\frac{1}{R}(k^2 + c) & 0 \\ jk & -\frac{1}{R}(k^2 + c) \end{bmatrix},$$

$$B^{\mathrm{o}}(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C^{\mathrm{o}}(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C(k) = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad L = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

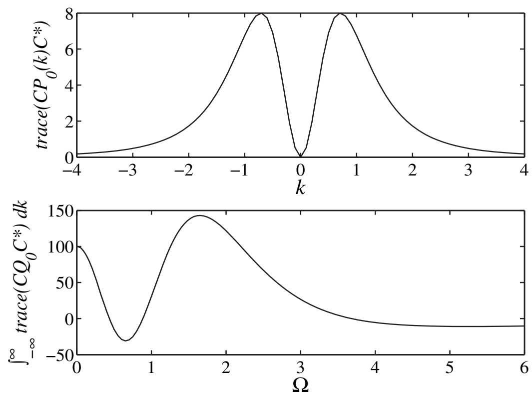

FIG. 7. *Plots of Example* 6 *for* $R = 6$ *and* $c = 1$.

Numerical calculations for $R = 6$, $c = 1$ give $\int_{-\infty}^{\infty} \text{trace}[C(k)P_0(k)C^*(k)]\,dk \approx 20.72$. Figure 7 shows that the $\mathcal{H}^2$ norm can be decreased by the application of periodic perturbations with frequency $\Omega \approx 0.7$. It is interesting that, if one uses the locations of the peaks in the first plot of Figure 7 to find $\varkappa = 0.75$, then from the peak at $\Omega_{\text{res}} = 1.6$ in the second plot it seems that the relationship $\Omega_{\text{res}} \approx 2\varkappa = 1.5$ (see Example 4) still holds with an acceptable error even for this matrix-valued system.

7. Stability and the spectrum-determined growth condition. In the literature on semigroups there exist examples in which $\Sigma(A)$ lies entirely inside $\mathbb{C}^-$ but $\|e^{At}\|$ does not decay exponetially; see [6] and more recently [7]. In such cases it is said that the semigroup does not satisfy the *spectrum-determined growth condition* [8]. The determining factor in the examples presented in [6, 7] is the accumulation of the eigenvalues of $\mathcal{A}_\theta$ around $\pm j\infty - c$, $c > 0$, in the form of Jordan blocks of ever-increasing size (i.e., as the eigenvalues tend to $\pm j\infty - c$, their algebraic multiplicity increases while their geometric multiplicity stays finite). But such cases are ruled out when one deals with semigroups generated by *sectorial* operators.

Our ultimate aim in this section is to verify exponential stability. From Theorem 2, in order to prove exponential stability of a holomorphic C_0 semigroup generated by a sectorial operator A, it is necessary and sufficient to verify that $\Sigma(A) \subset \mathbb{C}^-$. Hence in the first part of this section we give conditions under which the infinitesimal generator A in (10) is a sectorial operator. In the second part we find conditions which guarantee $\Sigma(A) \subset \mathbb{C}^-$.

Once again, the setup is that of (10) where ϵ is a small complex scalar. In addition, assume that $A_0(k) \in \mathbb{C}^{q\times q}$ is diagonalizable for every $k \in \mathbb{R}$.

Conditions for sectorial infinitesimal generator. To find conditions under which an infinitesimal generator A is sectorial, we have to verify (14), i.e., verify whether $\|(zI - A)^{-1}\| \leq M/|z - \alpha|$ for all z belonging to some sector of $\mathbb{C}$. This involves finding the inverse of the operator $zI - A$ and then calculating its norm. Such a computation can in general be very difficult. On the other hand, finding $\|(zI - A^{\text{o}})^{-1}\|$ is easy because of the spatial invariance of A^{o}. Indeed, from the norm-preserving property of the Fourier transform, it follows that $\|(zI - A^{\text{o}})^{-1}\| = \sup_{k\in\mathbb{R}} \|(zI - A_0(k))^{-1}\|$.

Thus to establish conditions for A to be sectorial we again use perturbation theory. We first find conditions under which A^{o} is sectorial. We then show that $A = A^{\text{o}} + \epsilon\, E$

remains sectorial if E is "weaker" than $A^{\rm o}$ in a certain sense we will describe and if ϵ is small enough.

In the next theorem we present a condition for a spatially invariant operator $A^{\rm o}$ with Fourier symbol $A_0(\,\cdot\,)$ to be sectorial.

THEOREM 5. *Let $A_0(k)$ be diagonalizable for every $k \in \mathbb{R}$, and let $U(k)$ be the transformation that diagonalizes $A_0(k)$, i.e., $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$, with $\Lambda(k)$ diagonal. Let $\kappa(k) := \|U(k)\|\,\|U^{-1}(k)\|$ denote the condition number of $U(k)$. If $\sup_{k\in\mathbb{R}} \kappa(k) < \infty$, and for every $k \in \mathbb{R}$ the resolvent set $\rho(A_0(k))$ contains a sector of the complex plane $|\arg(z-\alpha)| < \frac{\pi}{2} + \gamma$, with $\gamma > 0$ and $\alpha \in \mathbb{R}$ both independent of k, then A^o is sectorial.*

Proof. See the appendix. □

This theorem has a particularly simple interpretation when $A_0(\,\cdot\,)$ is scalar. In this case $\kappa(k) = 1$ for all $k \in \mathbb{R}$. Since $A_0(\,\cdot\,)$ traces a curve in the complex plane, by Theorem 5 if this curve stays outside some sector $|\arg(z-\alpha)| \le \frac{\pi}{2} + \gamma$, $\gamma > 0$, of the complex plane, then $A^{\rm o}$ is sectorial.

The following theorem is the main result of this section and uses the notion of *relative boundedness* of one unbounded operator with respect to another unbounded operator [9] to prove that $A = A^{\rm o} + \epsilon\, E$ is sectorial.

THEOREM 6. *Let A^o with domain $\mathscr{D}$ be a closed operator, and $A^o \in \mathscr{H}(\gamma, \alpha, M)$. Let $E = B^o\, F\, C^o$ with domain $\mathscr{D}' \supset \mathscr{D}$ be relatively bounded with respect to A^o such that*

$$\|E\,\psi\| \;\le\; a\,\|\psi\| \;+\; b\,\|A^o\,\psi\|, \quad \psi \in \mathscr{D}, \tag{34}$$

with $0 \le a < \infty$ and $0 \le b\,|\epsilon| < 1/(1+M)$. Then $A = A^o + \epsilon\, E$ is sectorial, closed, and generates a C_0 semigroup.

Proof. From (34) we have

$$\|\epsilon E\,\psi\| \;\le\; a\,|\epsilon|\,\|\psi\| \;+\; b\,|\epsilon|\,\|A^{\rm o}\,\psi\|.$$

Then from [11, Theorem 4.5.7] it follows that $A = A^{\rm o} + \epsilon\, E$ is sectorial for all ϵ such that $0 \le b\,|\epsilon| < 1/(1+M)$. Since $M > 0$ we have $b\,|\epsilon| < 1$, and [9, Theorem IV.1.1] gives that A is closed. Finally, it follows from [19, p. 100] that A is the generator of a C_0 semigroup. □

This theorem says that if $A^{\rm o}$ is sectorial and closed, then so is $A = A^{\rm o} + \epsilon\, E$ if E is weaker than $A^{\rm o}$ in the sense of (34) and if $|\epsilon|$ is small enough. Notice that at this point condition (34) cannot be reduced to a condition in terms of Fourier symbols as in Theorem 5 (i.e., a condition that can be checked pointwise in k). This is because E is not spatially invariant. But once the exact form of the operators $B^{\rm o}$ and $C^{\rm o}$ is known, (34) *can* be simplified to a condition on the Fourier symbols of $A^{\rm o}$, $B^{\rm o}$, and $C^{\rm o}$. Let us clarify this statement with the aid of an example.

Example 7. Consider the spatially periodic system

$$\begin{aligned} \partial_t\,\psi \;&=\; -\,(\partial_x^2 + \varkappa^2)^2\,\psi \,-\, c\,\psi \,+\, \epsilon\,\partial_x\,\cos(\Omega x)\,\psi \,+\, u, \\ y \;&=\; \psi, \end{aligned}$$

where $\psi \in \mathscr{D}$, and $\mathscr{D}$ is defined as in (17). It is easy to see that $A^{\rm o} = -(\partial_x^2 + \varkappa^2)^2 - c$ is sectorial by Theorem 5 and closed by Example 3, and $E = \partial_x \cos(\Omega x)$. By formal differentiation we have

$$E\,\psi \;=\; \partial_x\,\cos(\Omega x)\,\psi \;=\; -\,\Omega\,\sin(\Omega x)\,\psi \,+\, \cos(\Omega x)\,\partial_x\,\psi.$$

By using the triangle inequality and $\|\sin(\Omega x)\| = \|\cos(\Omega x)\| = 1$, we have

$$\|E\,\psi\| \;\leq\; |\Omega|\,\|\psi\| \,+\, \|\partial_x\,\psi\|. \tag{35}$$

Thus we have effectively commuted out the bounded spatially periodic component of E and are left with only spatially invariant operators on the right of (35). Hence after applying a Fourier transfomation to the right side of (35), a sufficient condition for (34) to hold is that

$$|\Omega| \,+\, |k| \;\leq\; a \,+\, b\,|(k^2 - \varkappa^2)^2 + c|, \qquad k \in \mathbb{R},$$

which can be shown to be satisfied for large enough $a > 0$ and $b > 0$. By using Theorem 6 we get that A is sectorial, closed, and the generator of a C_0 semigroup.

Remark 8. The above example makes clear the notion of E being "weaker" than A^{o} that we mentioned at the beginning of this subsection. If in Example 7 we had $B^{\mathrm{o}} = \partial_x^{\nu}$, $C^{\mathrm{o}} = \partial_x^{\mu}$, and $\nu + \mu = 5$, then E would contain a 5th-order derivative, whereas the highest order of ∂_x in A^{o} is 4. This would mean that (34) cannot be satisfied for any choice of a and b.

Conditions for infinitesimal generator with spectrum in $\mathbb{C}^-$. The final step in establishing exponential stability is to check whether $\Sigma(A) \subset \mathbb{C}^-$. Since this is, in general, a difficult problem, we proceed as follows. We consider the (block) diagonal operators $\mathcal{A}_\theta^{\mathrm{o}}$ and extend Geršgorin-type methods to find bounds on the location of $\Sigma(\mathcal{A}_\theta)$, $\mathcal{A}_\theta = \mathcal{A}_\theta^{\mathrm{o}} + \epsilon\,\mathcal{E}_\theta$. We then use this to find conditions on $\epsilon\,\mathcal{E}_\theta$ that yield $\Sigma(\mathcal{A}_\theta) \subset \mathbb{C}^-$ for all $\theta \in [0, \Omega)$.

In locating the spectrum of a finite-dimensional matrix $T \in \mathbb{C}^{q\times q}$, the theory of Geršgorin circles [28] provides us with a region of the complex plane that contains the eigenvalues of T. This region is composed of the union of q disks, the centers of which are the diagonal elements of T, and their radii depend on the magnitude of the off-diagonal elements [28]. This theory has also been extended to the case of finite-dimensional *block* matrices (i.e., matrices whose elements are themselves matrices) in [29]. We will further extend this theory to include biinfinite block matrices $\mathcal{A}_\theta$.

Take $\mathfrak{B}_k$ to be the set of complex numbers z that satisfy

$$\sigma_{\min}\big(zI - A_0(k)\big) \;\leq\; |\epsilon|\,\big(\|A_{-1}(k)\| + \|A_1(k)\|\big), \tag{36}$$

where $\sigma_{\min}(zI - A_0(k))$ denotes the smallest singular value of the matrix $zI - A_0(k)$.

Lemma 7. *For every θ we have $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta$, where*

$$\mathfrak{S}_\theta \;=\; \overline{\bigcup_{n\in\mathbb{Z}} \mathfrak{B}_{\theta_n}}.$$

Proof. See the appendix. ◻

Example 8. Let us consider again the spatially periodic system in Example 5,

$$\begin{aligned} \partial_t\,\psi \;&=\; -\,(\partial_x^2 + \varkappa^2)^2\,\psi \,-\, c\,\psi \,+\, \epsilon\,\cos(\Omega x)\,\partial_x\,\psi \,+\, u, \\ y \;&=\; \psi. \end{aligned}$$

From (33) and (13) we have $A_1(k) = \frac{j}{2}(k - \Omega)$, $A_{-1}(k) = \frac{j}{2}(k + \Omega)$, and thus $\|A_{-1}(k)\| + \|A_1(k)\| = \frac{1}{2}\left(|k - \Omega| + |k + \Omega|\right)$. Hence (36) leads to

$$\sigma_{\min}(zI - A_0(k)) \;=\; |zI - A_0(k)| \;\leq\; \frac{|\epsilon|}{2}\left(|k - \Omega| + |k + \Omega|\right) \;=\; \begin{cases} \Omega\,|\epsilon| & |k| \leq \Omega, \\ |k|\,|\epsilon| & |k| \geq \Omega, \end{cases}$$

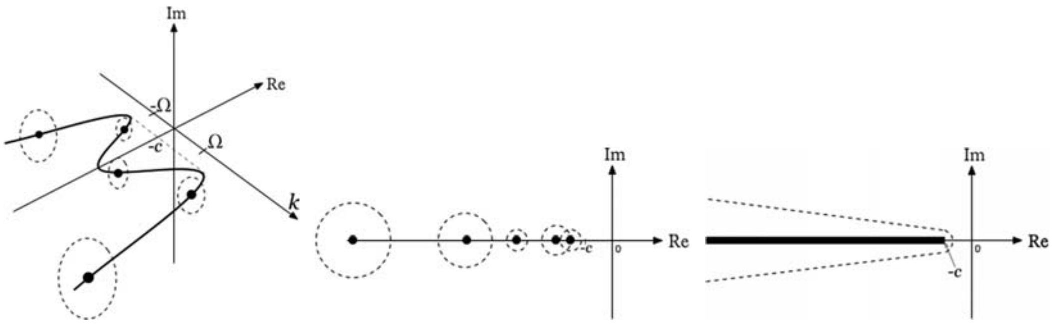

FIG. 8. *Left: The $\mathfrak{B}_{\theta_n}$ regions viewed in the complex-plane $\times$ spatial-frequency space (the disks are parallel to the complex plane). Center: $\Sigma(\mathcal{A}_\theta)$ is contained inside the union of the regions $\mathfrak{B}_{\theta_n}$. Right: The bold line shows $\Sigma(A^o)$, and the dotted region contains $\Sigma(A)$, $A = A^o + \epsilon E$.*

which means that the set $\mathfrak{S}_\theta$ is composed of the union of disks with centers at $A_0(\theta_n)$ and radii $\frac{|\epsilon|}{2}(|\theta_n - \Omega| + |\theta_n + \Omega|)$. This is nothing but an extension of the classical Geršgorin disks to biinfinite matrices. Figure 8 (left and center) show $\mathfrak{S}_\theta$ in the complex-plane $\times$ spatial-frequency space and in $\mathbb{C}$, respectively.[7]

Remark 9. The set

$$\begin{aligned} \Sigma_\varepsilon(T) &:= \{z \in \mathbb{C} \mid \sigma_{\min}(zI - T) \leq \varepsilon\} \\ &= \{z \in \mathbb{C} \mid \|(zI - T)\varphi\| \leq \varepsilon \text{ for some } \|\varphi\| = 1\} \\ &= \{z \in \mathbb{C} \mid z \in \Sigma_{\mathrm{p}}(T + Z) \text{ for some } \|Z\| \leq \varepsilon\} \end{aligned} \tag{37}$$

is called the *ε-pseudospectrum* of the matrix T [30, 31]. Clearly $\Sigma_{\varepsilon'}(T) \subseteq \Sigma_\varepsilon(T)$ if $\varepsilon' \leq \varepsilon$, and $\Sigma_\varepsilon(T) = \Sigma_{\mathrm{p}}(T)$ for $\varepsilon = 0$. The pseudospectrum is composed of small sets around the eigenvalues of T. For instance, if T has simple eigenvalues then for small enough values of ε the pseudospectrum consists of disjoint compact and convex neighborhoods of each eigenvalue [32]. By comparing (37) and the definition of $\mathfrak{B}_k$ in (36), it is clear that $\mathfrak{B}_k = \Sigma_\varepsilon(A_0(k))$, with $\varepsilon = |\epsilon|\,(\|A_{-1}(k)\| + \|A_1(k)\|)$. Thus for every $k \in \mathbb{R}$, the inequality (36) defines a closed region of $\mathbb{C}$ that includes the eigenvalues of $A_0(k)$.

We now employ Lemma 7 to determine whether $\Sigma(A)$ resides completely inside $\mathbb{C}^-$, as needed to assess system stability. Take $\mathfrak{D}_\varepsilon$ to be the closed disk of radius ε and center at the origin and $\mathfrak{B}_k$ to be the region described by (36). Define the sum of sets by $\mathfrak{U}_1 + \mathfrak{U}_2 = \{z \mid z = z_1 + z_2, z_1 \in \mathfrak{U}_1, z_1 \in \mathfrak{U}_1\}$. For every $k \in \mathbb{R}$ let $\lambda_{\max}(k)$ represent the eigenvalue of $A_0(k)$ with the maximum real part, and let $\kappa(k)$ be defined as in Theorem 5.

THEOREM 8. *For a given k the set $\mathfrak{B}_k$ is contained inside $\Sigma_p(A_0(k)) + \mathfrak{D}_{r(k)}$, with*

$$r(k) \;=\; |\epsilon|\left(\|A_{-1}(k)\| + \|A_1(k)\|\right)\kappa(k).$$

In particular, if $\Sigma(A^o) \subset \mathbb{C}^-$ and

$$r(k) \;<\; \left|\mathrm{Re}(\lambda_{\max}(k))\right| \;+\; \beta \tag{38}$$

for every $k \in \mathbb{R}$ and some $\beta < 0$ independent of k, then $\Sigma(A) \subset \mathbb{C}^-$.

[7]We point out that Figure 8 (left) is technically incorrect; once the spatially invariant system is perturbed by a spatially periodic perturbation, it is no longer spatially invariant and thus cannot be represented by a Fourier symbol. Hence its spectrum can no longer be demonstrated in the complex-plane $\times$ spatial-frequency space. Figure 8 (center) demonstrates the correct representation of the Geršgorin disks for $\mathcal{A}_\theta$.

Proof. See the appendix. □

Example 9. Consider the system of Example 8. $\kappa(k) = 1$ since $A_0(k)$ is scalar, $|\mathrm{Re}(\lambda_{\max}(k))| = |(k^2 - \varkappa^2)^2 + c|$, and

$$\|A_{-1}(k)\| + \|A_1(k)\| \;=\; \frac{1}{2}\,(|k - \Omega| + |k + \Omega|).$$

Thus condition (38) becomes

$$\frac{|\epsilon|}{2}\,(|k - \Omega| + |k + \Omega|) \;<\; |(k^2 - \varkappa^2)^2 + c| + \beta.$$

If this condition is satisfied for some $\beta < 0$, the dotted region in Figure 8 (right) will remain inside $\mathbb{C}^-$, and thus $\Sigma(A) \subset \mathbb{C}^-$.

In summary, to assess exponential stability we first find sufficient conditions on the infinitesimal generator A such that it belongs to the class of sectorial operators, for which the spectrum-determined growth condition holds. We then find sufficient conditions for A to have $\mathbb{C}^-$ spectrum. We do this via an extension of Geršgorin circles to biinfinite (block) matrices.

8. Conclusions and future work. We use perturbation analysis to find a computationally efficient way of revealing trends in the $\mathcal{H}^2$ norm of spatially periodic systems. We show that for certain classes of systems the periodicity can be chosen so as to increase the $\mathcal{H}^2$ norm and induce parametric resonance. An application of this would be in fluid mixing problems. It is also shown that the $\mathcal{H}^2$ norm can be made to decrease for an appropriate choice of the frequency of the perturbation. This would be the desired scenario, for example, in the design of the body of an aircraft. We demonstrate that for certain scalar systems the value of the spatial period that achieves the desired increase or decrease of the $\mathcal{H}^2$ norm can be characterized exactly based on the description of the nominal system.

We also study the problem of verifying the exponential stability of a spatially periodic system. We do this by (i) finding conditions under which its infinitesimal generator is a sectorial operator (i.e., generates a holomorphic C_0 semigroup) and thus satisfies the spectrum-determined growth condition and (ii) deriving conditions which guarantee that the infinitesimal generator has its spectrum contained inside the open left half of the complex plane.

The methods presented here can also be used in systems with many spatial directions. For example, consider the PDE

$$\psi_t \;=\; \psi_{yy} \,+\, \psi_{xx} \,+\, c\,\psi \,+\, \epsilon\,\cos(\Omega x)\,\psi,$$

with $y \in [-1, 1]$ and $x \in R$. To put this system into the developed framework one would only have to perform a discrete approximation of the operator ∂_y^2 with the appropriate boundary conditions. Furthermore, the techniques developed in this paper can be applied to spatially periodic systems defined on a torus with minor changes.

Future research in this direction would include an exact (analytical) characterization of the frequencies for which the $\mathcal{H}^2$ norm is most increased or decreased for the general case of matrix-valued $A_0(\,\cdot\,)$. The perturbation methods presented here could also be generalized to biinfinite Sylvester equations which arise frequently in fluids problems.

Appendix.

Convergence of the perturbation series. For the series expansion

$$\mathcal{P}_\theta(\epsilon) \;=\; \mathcal{P}_\theta^{(0)} \,+\, \epsilon\,\mathcal{P}_\theta^{(1)} \,+\, \epsilon^2\,\mathcal{P}_\theta^{(2)} \,+\, \cdots$$

to be valid, we must show that all elements of the biinfinite matrix $\mathcal{P}_\theta(\epsilon)$ converge for ϵ contained in some small enough neighborhood of the origin. Let us assume that B^{o} and C^{o} are bounded operators and (without loss of generality) that $\sup_{k\in\mathbb{R}} \|B(k)\| = 1$. Assume that $\|e^{A_0(k)t}\| \le M_k\, e^{\varrho_k t}$, and define

$$\begin{aligned} M \;&:=\; \sup_{k\in\mathbb{R}} M_k \;<\; \infty, \qquad -\,\alpha_0 \;:=\; \sup_{k\in\mathbb{R}} \varrho_k \;<\; 0, \\ \alpha_1 \;&:=\; \max\{\sup_{k\in\mathbb{R}} \|A_1(k)\|,\, \sup_{k\in\mathbb{R}} \|A_{-1}(k)\|\}. \end{aligned}$$

Notice that the finiteness of M and the negativity of $-\alpha_0$ follow from the exponential stability of the unperturbed system. Now from (23) we have

$$P_0(k) \;=\; \int_0^\infty e^{A_0(k)t} B(k)B^*(k)\, e^{A_0^*(k)t}\, dt,$$

and therefore

$$\sup_{k\in\mathbb{R}} \|P_0(k)\| \;\le\; \frac{M^2}{2\alpha_0} \;=:\; \mu.$$

Similarly, from (24) we have

$$P_1(k) \;=\; \int_0^\infty e^{A_0(k)t}\big(A_1(k)P_0(k-\Omega) + P_0(k)A_{-1}^*(k-\Omega)\big)\, e^{A_0^*(k-\Omega)t}\, dt,$$

and therefore

$$\sup_{k\in\mathbb{R}} \|P_1(k)\| \;\le\; \frac{M^2}{2\alpha_0}(2\alpha_1\mu) \;\le\; \mu(4\,\alpha_1\mu) \;=\; 4\,\alpha_1\mu^2.$$

From (25) we have

$$Q_0(k) \;=\; \int_0^\infty e^{A_0(k)t}\big(A_1(k)P_1^*(k) + \cdots + P_1^*(k+\Omega)A_{-1}^*(k)\big)\, e^{A_0^*(k)t}\, dt,$$

and therefore

$$\sup_{k\in\mathbb{R}} \|Q_0(k)\| \;\le\; \frac{M^2}{2\alpha_0}(4\,\alpha_1(4\,\alpha_1\mu^2)) \;\le\; \mu(4\,\alpha_1)^2\mu^2 \;=\; (4\,\alpha_1)^2\mu^3.$$

In fact it is possible to show that any element of $\mathcal{P}_\theta^{(m)}$ is bounded by

$$(4\,\alpha_1)^m\mu^{(m+1)}.$$

Thus for all $|\epsilon| < 4\,\alpha_1\mu = 2M^2\alpha_1/\alpha_0$ the series expansion of $\mathcal{P}_\theta(\epsilon)$ converges.

Proof of Theorem 5. Recall that A^{o} is a sectorial operator if $\rho(A^{\mathrm{o}})$ contains a (right) sector of the complex plane $|\arg(z-\alpha)| \le \frac{\pi}{2} + \gamma$, $\gamma > 0$, $\alpha \in \mathbb{R}$, and there exists some $M > 0$ such that

$$|z-\alpha|\; \|(zI - A^{\mathrm{o}})^{-1}\| \;\le\; M \quad \text{for} \quad |\arg(z-\alpha)| \;\le\; \frac{\pi}{2} + \gamma.$$

Since $A_0(k) \in \mathbb{C}^{q\times q}$ is diagonalizable for every k, there exists a matrix $U(k)$ such that $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$, with $\Lambda(k)$ a diagonal matrix. Let $\lambda_i(k)$, $i = 1,\dots,q$, denote the diagonal elements of $\Lambda(k)$, which are also the eigenvalues of $A_0(k)$. Then

$$\begin{aligned} |z-\alpha|\;\|(zI - A^{\rm o})^{-1}\| \;&\le\; \sup_{k\in\mathbb{R}} \Big(|z-\alpha|\;\|(zI - A_0(k))^{-1}\| \Big) \\ &\le\; \sup_{k\in\mathbb{R}} \Big(|z-\alpha|\;\|U(k)\|\;\|U^{-1}(k)\|\;\|(zI-\Lambda(k))^{-1}\| \Big) \\ &=\; \sup_{k\in\mathbb{R}} \left(\kappa(k)\;\frac{|z-\alpha|}{\mathrm{dist}[z,\Sigma_{\rm p}(A_0(k))]} \right) \\ &\le\; \kappa_{\max}\;\sup_{k\in\mathbb{R}} \left(\frac{|z-\alpha|}{\mathrm{dist}[z,\Sigma_{\rm p}(A_0(k))]} \right), \end{aligned}$$

where $\kappa_{\max} := \sup_{k\in\mathbb{R}} \kappa(k)$.

Set $M = (M'+1)\,\kappa_{\max}$, with $M' > 0$. Consider for a given k the region of the complex plane defined by

$$\kappa_{\max}\;\frac{|z-\alpha|}{\mathrm{dist}[z,\Sigma_{\rm p}(A_0(k))]} \;\ge\; M.$$

This region (which contains the eigenvalues $\lambda_i(k)$) is contained inside the union of the disks

$$\kappa_{\max}\;\frac{|z-\alpha|}{|z-\lambda_i(k)|} \;\ge\; M, \qquad i = 1,\dots,q,$$

which are themselves contained inside the larger disks

$$|z-\lambda_i(k)| \;\le\; \frac{|\lambda_i(k)-\alpha|}{M'}, \qquad i = 1,\dots,q. \tag{A1}$$

Notice that (A1) describes disks whose radii increase like $|\lambda_i(k)-\alpha|/M'$, $M' > 0$, as their centers $\lambda_i(k)$ grow distant from the point α. A sufficient condition for these disks to belong to some open (left) sector of the complex plane $|\arg(z-\alpha)| > \frac{\pi}{2}+\gamma$, $\gamma > 0$, for all $k \in \mathbb{R}$ and large enough M' is that $\Sigma_{\rm p}(A_0(k))$, $k\in\mathbb{R}$, reside inside some open (left) sector of the complex plane $|\arg(z-\alpha)| > \frac{\pi}{2}+\gamma'$, $\gamma' > \gamma$.

Finally, if the conditions of the previous paragraph are satisfied then for $z \in \mathbb{C}$ that belong to the sector $|\arg(z-\alpha)| \le \frac{\pi}{2}+\gamma$ we have $z \in \rho(A_0(k))$ and

$$\kappa_{\max}\;\sup_{k\in\mathbb{R}} \left(\frac{|z-\alpha|}{\mathrm{dist}[z,\Sigma_{\rm p}(A_0(k))]} \right) \;\le\; M.$$

Thus $|z-\alpha|\;\|(zI-A^{\rm o})^{-1}\| \le M$, and $A^{\rm o}$ is sectorial. □

Proof of Lemma 7. We use $\Pi_N\,T\,\Pi_N$ to denote the $(2N+1)\times(2N+1)$ [block] truncation of an operator T on ℓ^2, where Π_N is the projection defined by

$$\mathrm{diag}\Big\{\dots,0,\underbrace{I,\dots,\overset{\substack{\textbf{center}\\ \downarrow}}{I},\;\dots,I}_{2N+1\ \text{times}},0,\dots\Big\},$$

and I is the identity matrix. Notice that $\Pi_N\,T\,\Pi_N$ is still an operator on ℓ^2; it is made from the biinfinite matrix T by replacing all entries outside the center $(2N{+}1)\times(2N{+}1)$

block with zeros. We now form the finite-dimensional matrix $\Pi_N\,\mathcal{A}_\theta\,\Pi_N\big|_{\Pi_N \ell^2}$ by restricting $\Pi_N\,\mathcal{A}_\theta\,\Pi_N$ to the finite-dimensional space $\Pi_N \ell^2$. Clearly $\Pi_N\,\mathcal{A}_\theta\,\Pi_N\big|_{\Pi_N \ell^2}$ has pure point spectrum. Hence, using a generalized form of the Geršgorin circle theorem [29] for finite-dimensional (block) matrices, we conclude that

$$\Sigma\big(\Pi_N\,\mathcal{A}_\theta\,\Pi_N\big|_{\Pi_N \ell^2}\big) \;\subset\; \bigcup_{|n|\leq N} \mathfrak{B}_{\theta_n} \;\subseteq\; \overline{\bigcup_{n\in\mathbb{Z}} \mathfrak{B}_{\theta_n}},$$

where $\mathfrak{B}_{\theta_n}$ are regions of $\mathbb{C}$ defined by (36). Since this holds for all $N \geq 0$, we have $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta$. □

Proof of Theorem 8. If $U(k)$ diagonalizes $A_0(k)$, $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$, and $\kappa(k) = \|U(k)\|\,\|U^{-1}(k)\|$ denotes the condition number of $U(k)$, then from [33] the pseudospectrum of $A_0(k)$ satisfies

$$\Sigma_{\rm p}(A_0(k)) \,+\, \mathfrak{D}_\varepsilon \;\subseteq\; \Sigma_\varepsilon(A_0(k)) \;\subseteq\; \Sigma_{\rm p}(A_0(k)) \,+\, \mathfrak{D}_{\varepsilon\kappa(k)} \tag{A2}$$

for all $\varepsilon \geq 0$. Thus the first statement of the theorem follows immediately from (A2) and the fact that $\mathfrak{B}_k = \Sigma_\varepsilon(A_0(k))$, with $\varepsilon = |\epsilon|\,(\|A_{-1}(k)\| + \|A_1(k)\|)$ [see Remark 9].

To prove the second statement, let $\mathbb{C}^-_\beta$ denote all complex numbers with a real part less than $\beta \in \mathbb{R}$. It follows from $\Sigma(A^{\rm o}) \subset \mathbb{C}^-$ that $\Sigma(\mathcal{A}^{\rm o}_\theta) \subset \mathbb{C}^-$ for every θ. If (38) holds, then

$$\mathfrak{B}_{\theta_n} \;\subseteq\; \Sigma_{\rm p}(A_0(\theta_n)) \,+\, \mathfrak{D}_{r(\theta_n)} \;\subset\; \mathbb{C}^-_\beta$$

for every $n \in \mathbb{Z}$, and from Lemma 7 we have $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta = \overline{\bigcup_{n\in\mathbb{Z}} \mathfrak{B}_{\theta_n}} \subset \mathbb{C}^-_{\beta'}$ for some $\beta < \beta' < 0$ and every θ. Thus $\Sigma(A) \subset \mathbb{C}^-$.

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