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An Extension of the Argument Principle and Nyquist Criterion to a Class of Systems With Unbounded Generators

Makan Fardad and Bassam Bamieh

***Abstract*—The Nyquist stability criterion is generalized to systems where the open-loop system has infinite-dimensional input and output spaces and an unbounded infinitesimal generator. The infinitesimal generator is assumed to be a sectorial operator with trace-class resolvent. The main result is obtained through use of the perturbation determinant and an extension of the argument principle to infinitesimal generators with trace-class resolvents.**

***Index Terms*—Argument principle, infinite-dimensional system, Nyquist stability criterion, perturbation determinant, unbounded infinitesimal generator.**

I. Introduction

The Nyquist criterion is of particular interest in system analysis as it offers a simple visual test to determine the stability of a closed-loop system for a family of feedback gains [1], [2]. Extensions of the Nyquist stability criterion exist for certain classes of distributed [3] and time-periodic [4] systems. Desoer and Wang [3] consider distributed systems in which the open-loop transfer function $G(s)$ belongs to the algebra of matrix-valued meromorphic functions of *finite* Euclidean dimension, and the Nyquist analysis is carried out by performing a coprime factorization on $G(s)$.

To motivate the discussion in this paper, let us first consider a finite-dimensional (multiinput multioutput) LTI system $G(s)$ placed in feedback with a constant gain γI. In analyzing the closed-loop stability of such a system, we are concerned with the eigenvalues in $\mathbb{C}^+$ of the closed-loop dynamics A^{cl}. If s is an eigenvalue of A^{cl}, then it satisfies $\det[sI - A^{\mathrm{cl}}] = 0$. Now to check whether the equation $\det[sI - A^{\mathrm{cl}}] = 0$ has solutions inside $\mathbb{C}^+$, one can apply the argument principle to $\det[I + \gamma G(s)]$ as s traverses some path $\mathfrak{D}$ enclosing $\mathbb{C}^+$. To elaborate, let us assume that we are given a state-space realization of the open-loop system. Then, using

$$\det[I + \gamma G(s)] = \frac{\det[sI - A^{\mathrm{cl}}]}{\det[sI - A]} \tag{1}$$

if one knows the number of unstable open-loop poles, one can determine the number of unstable closed-loop poles by looking at the plot of $\det[I + \gamma G(s)]\big|_{s\in\mathfrak{D}}$. But in the case of distributed systems, the open-loop and closed-loop infinitesimal generators $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$ are operators on an infinite-dimensional Hilbert space $\mathcal{X}$ and can be *unbounded*. Hence, it is not clear how to define the characteristic functions $\det[s\mathcal{I} - \mathcal{A}]$ and $\det[s\mathcal{I} - \mathcal{A}^{\mathrm{cl}}]$. In this paper, we find an analog of (1) applicable to unbounded $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$ and use operator-theoretic arguments to relate

Manuscript received May 8, 2006; revised December 11, 2006 and June 28, 2007. Recommended by Associate Editor D. Dochain. This work was supported in part by the Air Force Office of Scientific Research under Grant FA9550-04-1-0207 and in part by the National Science Foundation under Grant ECS-0323814.

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Digital Object Identifier 10.1109/TAC.2007.914233

the plot of $\det[\mathcal{I}+\gamma\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}$ to the unstable modes of the open-loop and closed-loop systems.

If the multiplicity of each of the eigenvalues of $\mathcal{A}$ is finite, it can be shown that $\det[\mathcal{I}+\gamma\mathcal{G}(s)]$ is a meromorphic function of s on $\mathbb{C}$, and one may be tempted to use the methods of [3] to analyze closed-loop stability. But if the open-loop system has distributed input and output spaces, then application of the framework of [3] would require the coprime factorization of an *infinite-dimensional* operator. Furthermore, in dealing with infinite-dimensional systems, one is often faced with partial differential equations (PDEs) in which the state-space representation is the natural representation and it is more convenient to work directly with the operators $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$ rather than with the transfer operator $\mathcal{G}(s)$ [see example in Section IV].

Our presentation is organized as follows: We lay out the problem setup in Section II and describe the general conditions for stability of distributed systems. Section III contains the main contributions of the paper in which the argument principle and the Nyquist stability criterion are extended to a class of distributed systems. The theory is applied to a simple example in Section IV. Proofs and technical details have been placed in the Appendix.

A. Notation

$\Sigma(T)$ is the spectrum of the operator T, and $\rho(T)$ its resolvent set. $\sigma_n(T)$ is the nth singular-value of T. $B(\mathfrak{X})$ denotes the bounded operators on the Hilbert space $\mathfrak{X}$, $B_\infty(\mathfrak{X})$ the compact operators on $\mathfrak{X}$, and $B_1(\mathfrak{X})$ the nuclear (trace-class) operators on $\mathfrak{X}$, i.e., operators T that have the property $\sum_{n=1}^{\infty}\sigma_n(T)<\infty$; $B_1(\mathfrak{X})\subset B_\infty(\mathfrak{X})\subset B(\mathfrak{X})$. $\mathrm{tr}[T]$ denotes the trace of T and $\det[T]$ its determinant. $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the *closed* right-half and the *open* left-half of the complex plane, respectively, and $j:=\sqrt{-1}$. $C(z_0;\mathfrak{P})$ is the number of counterclockwise encirclements of the point $z_0\in\mathbb{C}$ by the closed path $\mathfrak{P}$, and $\angle z$ is the phase of the complex number z.

II. Problem Setup and Exponential Stability

Consider the open-loop system $\mathbf{S}^{\mathrm{o}}$ of the form

$$\begin{aligned}[\partial_t\psi](t)&=[\mathcal{A}\psi](t)+[\mathcal{B}u](t)\\ y(t)&=[\mathcal{C}\psi](t)\end{aligned}\tag{2}$$

with $t\in[0,\infty)$ and the following assumptions. At any given point t in time, the distributed state ψ, the input u, and the output y belong to the spaces $\mathfrak{X}$, $\mathfrak{U}$, and $\mathfrak{Y}$, respectively. We assume that $\mathfrak{U}=\mathfrak{Y}$, and $\mathfrak{X}$, $\mathfrak{Y}$ can be finite- or infinite-dimensional Hilbert spaces. The (possibly unbounded) operator $\mathcal{A}$ is defined on a dense domain $D(\mathcal{A})$ of the Hilbert space $\mathfrak{X}$, is closed, and generates a *strongly continuous* semigroup (also known as C_0 semigroup) denoted by $e^{\mathcal{A}t}$ [5]. The operators $\mathcal{B}$ and $\mathcal{C}$ are bounded. We will refer to $\mathcal{A}$ as the *infinitesimal generator* of the system. We may also refer to $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ as the *system operators*. The open-loop system $\mathbf{S}^{\mathrm{o}}$ has a temporal impulse response $\mathcal{G}(t)=\mathcal{C}e^{\mathcal{A}t}\mathcal{B}$, and a transfer function

$$\mathcal{G}(s)=\mathcal{C}(s\mathcal{I}-\mathcal{A})^{-1}\mathcal{B}.\tag{3}$$

We also make the following assumptions on the infinitesimal generator $\mathcal{A}$.

Assumption $(*)$: The operator $\mathcal{A}$ is sectorial [6], [7], i.e., its resolvent set $\rho(\mathcal{A})$ contains a sector of the complex plane $|\arg(z-\alpha)|\le\frac{\pi}{2}+\varphi$, $\varphi>0$, $\alpha\in\mathbb{R}$, and there exists some $M>0$ such that

$$\|(zI-\mathcal{A})^{-1}\|\le\frac{M}{|z-\alpha|}\quad\text{for } |\arg(z-\alpha)|\le\frac{\pi}{2}+\varphi.$$

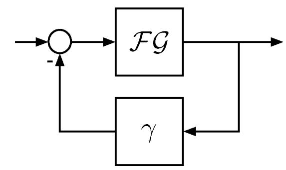

Fig. 1. Closed-loop system $\mathbf{S}^{\mathrm{cl}}$ in the standard form for Nyquist stability analysis.

This condition implies that the semigroup generated by $\mathcal{A}$ is analytic.

Assumption $(**)$: There exists at least one $s\in\rho(\mathcal{A})$ such that $(s\mathcal{I}-\mathcal{A})^{-1}\in B_1(\mathfrak{X})$.

Next, we place the system $\mathbf{S}^{\circ}$ in feedback with a bounded operator $\gamma\mathcal{F}$, $\|\mathcal{F}\|=1$, $\gamma\in\mathbb{C}$, which acts on the space $\mathfrak{Y}$. This forms the closed-loop system $\mathbf{S}^{\mathrm{cl}}$ shown in Fig. 1 with infinitesimal generator $\mathcal{A}^{\mathrm{cl}}:=\mathcal{A}-\gamma\mathcal{B}\mathcal{F}\mathcal{C}$. Our aim is to determine the exponential stability of $\mathbf{S}^{\mathrm{cl}}$ as the feedback gain γ varies in $\mathbb{C}$.

A semigroup $e^{\mathcal{A}t}$ on a Hilbert space is called *exponentially stable* if there exist constants $M\ge1$ and $\beta>0$ such that [5]

$$\|e^{\mathcal{A}t}\|\le Me^{-\beta t}\quad\text{for } t\ge0.$$

It is well-known [8], [9] that if $\mathcal{A}$ is an infinite-dimensional operator, then, in general, the condition

$$\sup_{z\in\Sigma(\mathcal{A})}\mathrm{Re}(z)<0\tag{4}$$

is *not* sufficient to guarantee exponential stability. Operators for which inequality (4) *does* imply exponential stability are said to satisfy the *spectrum-determined growth condition*. Sectorial operators have the important property that they satisfy the spectrum-determined growth condition.

Theorem 1: Under Assumption $(*)$, the closed-loop system $\mathbf{S}^{\mathrm{cl}}$ is exponentially stable if and only if $\Sigma(\mathcal{A}^{\mathrm{cl}})\subset\mathbb{C}^-$.

Proof: See Appendix.

III. The Nyquist Stability Criterion for Distributed Systems

In this section, we aim to develop a graphical method of checking closed-loop stability. Henceforth, in this paper, wherever we use the term stability, we mean exponential stability. For simplicity we absorb the operator $\mathcal{F}$ into $\mathcal{C}$ and introduce

$$\widetilde{\mathcal{C}}:=\mathcal{F}\mathcal{C},\quad \widetilde{\mathcal{G}}(s):=\widetilde{\mathcal{C}}(s\mathcal{I}-\mathcal{A})^{-1}\mathcal{B}.$$

A. The Determinant Method

As discussed in the introduction, we aim to use operator-theoretic arguments to relate the plot of $\det[\mathcal{I}+\gamma\widetilde{\mathcal{G}}(s)]\big|_{s\in\mathfrak{D}}$ to the unstable modes of the open-loop and closed-loop systems. But first, it has to be clarified what is meant by $\det[\mathcal{I}+\gamma\widetilde{\mathcal{G}}(s)]$ when $\mathcal{I}+\gamma\widetilde{\mathcal{G}}(s)$ is an infinite-dimensional operator.

From Assumption $(**)$ we know that $(s\mathcal{I}-\mathcal{A})^{-1}\in B_1(\mathfrak{X})$ for some $s\in\rho(\mathcal{A})$. Then, it is simple to show that $(s\mathcal{I}-\mathcal{A})^{-1}\in B_1(\mathfrak{X})$ *for every* $s\in\rho(\mathcal{A})$ [10]. Furthermore, from the boundedness of the operators $\mathcal{B}$ and $\widetilde{\mathcal{C}}=\mathcal{F}\mathcal{C}$, we get that $\widetilde{\mathcal{G}}(s)\in B_1(\mathfrak{Y})$. We can now define [10], [11]

$$\det[\mathcal{I}+\gamma\widetilde{\mathcal{G}}(s)]:=\prod_{n\in\mathbb{Z}}\big(1+\gamma\lambda_n(s)\big)$$

where $\{\lambda_n(s)\}_{n\in\mathbb{Z}}$ are the eigenvalues of $\widetilde{\mathcal{G}}(s)$.

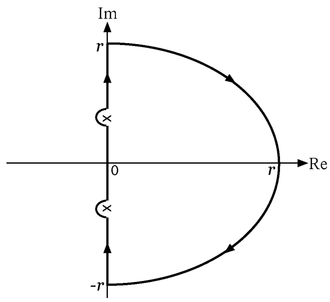

Fig. 2. Closed contour $\mathfrak{D}$ traversed in the clockwise direction taken as the Nyquist path as $r \to \infty$. The indentations are made to avoid the eigenvalues of $\mathcal{A}$ (i.e., open-loop modes) on the imaginary axis.

On the other hand, the boundedness of the operators $\mathcal{B}$ and $\widetilde{\mathcal{C}}$ together with Assumption ($**$) imply that 1) the operators $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}} = \mathcal{A} - \gamma\mathcal{B}\widetilde{\mathcal{C}}$ are defined on the same dense domain $D(\mathcal{A})$; 2) the set $\rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\mathrm{cl}})$ is not empty; and 3) for all $s \in \rho(\mathcal{A})$ we have $\gamma\mathcal{B}\widetilde{\mathcal{C}}(s\mathcal{I} - \mathcal{A})^{-1} \in B_1(\mathfrak{X})$. This allows us to introduce the *perturbation determinant* [12]

$$\begin{aligned}\Delta_{\mathcal{A}^{\mathrm{cl}}/\mathcal{A}}(s) &:= \det[(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})(s\mathcal{I} - \mathcal{A})^{-1}] \\ &= \det[\mathcal{I} + \gamma\mathcal{B}\widetilde{\mathcal{C}}(s\mathcal{I} - \mathcal{A})^{-1}] \\ &= \det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]\end{aligned}$$

which is analytic in $\rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\mathrm{cl}})$ [see Lemma A1]. In fact, $\Delta_{\mathcal{A}^{\mathrm{cl}}/\mathcal{A}}(s)$ is the equivalent of the fraction in (1) for systems with unbounded infinitesimal generators. We are now ready to state an extended form of the argument principle for such systems. The following theorem makes use of the formula [12]

$$\frac{d}{ds}\ln \Delta_{\mathcal{A}^{\mathrm{cl}}/\mathcal{A}}(s) = \mathrm{tr}[(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1} - (s\mathcal{I} - \mathcal{A})^{-1}] \quad \text{for all } s \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\mathrm{cl}}) \tag{5}$$

to relate $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]\big|_{s\in\mathfrak{D}}$ to the eigenvalues of $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$ inside the Nyquist path $\mathfrak{D}$.

Theorem 2: If $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)] \neq 0$ for all $s \in \mathfrak{D}$,

$$\begin{aligned}&C\Big(0; \det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]\big|_{s\in\mathfrak{D}}\Big) \\ &= \mathrm{tr}\left[\frac{1}{2\pi j}\int_{\mathfrak{D}}(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1}ds\right] - \mathrm{tr}\left[\frac{1}{2\pi j}\int_{\mathfrak{D}}(s\mathcal{I} - \mathcal{A})^{-1}ds\right] \\ &= -(\text{number of eigenvalues of } \mathcal{A}^{\mathrm{cl}} \text{ in } \mathbb{C}^+) \\ &\quad + (\text{number of eigenvalues of } \mathcal{A} \text{ in } \mathbb{C}^+)\end{aligned}$$

where $\mathfrak{D}$ is the Nyquist path in Fig. 2 that does not pass through any eigenvalues of $\mathcal{A}$.

Proof: See Appendix.

Remark 1: Theorem 2 relies on the fact that under Assumption ($**$) both $(s\mathcal{I} - \mathcal{A})^{-1}$ and $(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1}$ are compact operators, which implies that the infinitesimal generators $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$ have discrete spectrum (i.e., their spectrum consists entirely of isolated eigenvalues with finite multiplicity). Then $\mathcal{P} = -\frac{1}{2\pi j}\int_{\mathfrak{D}}(s\mathcal{I} - \mathcal{A})^{-1}ds$ is the group-projection [10], [13] corresponding to the eigenvalues of $\mathcal{A}$ inside $\mathfrak{D}$, and $\mathrm{tr}[\mathcal{P}]$ gives the total number of such eigenvalues [6]. Similarly $\mathrm{tr}[-\frac{1}{2\pi j}\int_{\mathfrak{D}}(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1}ds]$ gives the total number of eigenvalues of $\mathcal{A}^{\mathrm{cl}}$ in $\mathfrak{D}$.

As a direct consequence of Theorem 2, we have the following.

Theorem 3: Assume p_+ denotes the number of eigenvalues of $\mathcal{A}$ inside $\mathbb{C}^+$. For $\mathfrak{D}$ taken as earlier, the closed-loop system is stable iff

1) $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)] \neq 0, \forall s \in \mathfrak{D}$,

and

2) $C\Big(0; \det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]\big|_{s\in\mathfrak{D}}\Big) = p_+$. ■

B. The Eigenloci Method

The setback with the method developed earlier is that to show $\Sigma(\mathcal{A}^{\mathrm{cl}}) \subset \mathbb{C}^-$, $\mathcal{A}^{\mathrm{cl}} = \mathcal{A} - \gamma\mathcal{B}\mathcal{F}\mathcal{C}$ for different values of γ, one has to plot $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]\big|_{s\in\mathfrak{D}}$ for each γ. Note that this includes having to calculate the determinant of an infinite-dimensional matrix. This motivates the following eigenloci approach to Nyquist stability analysis, which is very similar to that performed in [4] for the case of time-periodic systems.

Let $\{\lambda_n(s)\}_{n\in\mathbb{Z}}$ constitute the eigenvalues of $\widetilde{\mathcal{G}}(s)$. Then

$$\angle\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)] = \angle\prod_{n\in\mathbb{Z}}\big(1 + \gamma\lambda_n(s)\big). \tag{6}$$

Recall that $\widetilde{\mathcal{G}}(s) \in B_1(\mathfrak{Y})$ for every $s \in \rho(\mathcal{A})$. This, in particular, means that $\widetilde{\mathcal{G}}(s)$ is a compact operator, and thus, its eigenvalues $\lambda_n(s)$ accumulate at the origin as $|n| \to \infty$ [14]. As a matter of fact, one can make a much stronger statement.

Lemma 4: The eigenvalues $\lambda_n(s)$, $s \in \mathfrak{D}$ converge to the origin uniformly on $\mathfrak{D}$.

Proof: See Appendix. ■

Take the positive integer N_ϵ to be such that $|\lambda_n(s)| < \epsilon$, $s \in \mathfrak{D}$ for all $|n| > N_\epsilon$. Let us rewrite (6) as

$$\begin{aligned}&\angle\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)] \\ &= \angle\prod_{|n|\le N_\epsilon}\big(1 + \gamma\lambda_n(s)\big) + \angle\prod_{|n|> N_\epsilon}\big(1 + \gamma\lambda_n(s)\big) \\ &= \sum_{|n|\le N_\epsilon}\angle\big(1 + \gamma\lambda_n(s)\big) + \sum_{|n|> N_\epsilon}\angle\big(1 + \gamma\lambda_n(s)\big).\end{aligned} \tag{7}$$

It is clear that if $|\gamma| < 1/\epsilon$, then for $|n| > N_\epsilon$, we have $|\gamma\lambda_n(s)| < 1$ and $1 + \gamma\lambda_n(s)$ can never circle the origin as s travels around $\mathfrak{D}$. Thus, for $|1/\gamma| > \epsilon$, the final sum in (7) will not contribute to the encirclements of the origin, and hence, we lose nothing by considering only the first N_ϵ eigenvalues. There still remain some minor technicalities.

First, let D_ϵ denote the disk $|s| < \epsilon$ in the complex plane. Then, the previous truncation may result in some eigenloci (parts of which reside inside D_ϵ) not forming closed loops. But notice that these can be arbitrarily closed inside D_ϵ as this does not affect the encirclements [4].

The second issue is that for some values of $s \in \mathfrak{D}$, the operator $\widetilde{\mathcal{G}}(s)$ may have multiple eigenvalues, and hence, there is ambiguity in how the eigenloci of the Nyquist diagram should be indexed. But this poses no problem as far as counting the encirclements is concerned and it is always possible to find such an indexing; for a detailed treatment see [3].

Let us denote by $\{\lambda_n\}_{n\in\mathbb{Z}}$ the indexed eigenloci that make up the generalized Nyquist diagram. [To avoid confusion, we stress the notation: $\lambda_n(s)$ is the nth *eigenvalue* of $\widetilde{\mathcal{G}}(s)$ for a given point $s \in \mathfrak{D}$, whereas λ_n is the nth *eigenlocus* traced out by $\lambda_n(s)$ as s travels once around $\mathfrak{D}$.] From (7) and the above discussion it follows that

$$C\Big(0; \det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]\big|_{s\in\mathfrak{D}}\Big) = \sum_{|n|\le N_\epsilon} C\Big(-\frac{1}{\gamma}; \lambda_n\Big)$$

which together with Theorem 3 gives the following.

Theorem 5: Assume p_+ denotes the number of eigenvalues of $\mathcal{A}$ inside $\mathbb{C}^+$. For $\mathfrak{D}$ and N_ϵ as defined previously, the closed-loop system is stable for $|1/\gamma| > \epsilon$ iff

1) $-\frac{1}{\gamma} \notin \{\lambda_n\}_{|n|\le N_\epsilon}$,

and

2) $\sum_{|n|\le N_\epsilon} C\left(-\frac{1}{\gamma};\lambda_n\right) = p_+.$

IV. An Illustrative Example

Consider the system defined on the spatial domain $x \in [0, 2\pi]$ and governed by the PDE

$$\partial_t \psi(t,x) = \partial_{xx}\psi(t,x) - \gamma\cos(x)\psi(t,x) + \psi(t,x),$$

with the periodic boundary conditions

$$\psi(t,0) = \psi(t,2\pi), \quad \partial_x\psi(t,0) = \partial_x\psi(t,2\pi).$$

This system can be thought of as describing heat propagation on a ring. It is possible to show that this PDE is unstable for $\gamma = 0$. We would like to find an answer to the following question: Does there exist any value of $\gamma \in \mathbb{R}$ for which this system is stable?[1]

Let us rewrite this system in the form of a PDE with constant coefficients described by

$$\begin{aligned}\partial_t \psi(t,x) &= \partial_{xx}\psi(t,x) + \psi(t,x) + u(t,x)\\ y(t,x) &= \psi(t,x)\end{aligned} \tag{8}$$

placed in feedback with the spatially periodic function

$$\gamma F(x) = \gamma\cos(x).$$

The problem is now in the general form discussed in Section II on the Hilbert space $X = L^2[0,2\pi]$. Clearly $A = \partial_{xx} + 1$ and is defined on the dense domain

$$D(A) = \{\phi \in L^2[0,2\pi] \mid \phi, \frac{d\phi}{dx} \text{ absolutely continuous}, \frac{d^2\phi}{dx^2} \in L^2[0,2\pi], \phi(0) = \phi(2\pi), \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi)\}.$$

The input and output operators are the identity, and $F = \cos(x)$.

We take an extra step and use a similarity transformation to put the problem in an *equivalent* form that is more familiar to us from multivariable linear systems theory. The Fourier series is a unitary transformation that takes a function $\phi(x) \in L^2[0,2\pi]$, $\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n e^{jnx}$, to the column vector $\mathrm{col}[\cdots,\phi_{-1},\phi_0,\phi_1,\cdots] \in \ell^2$. We apply this transformation to all spatial functions and operators in the PDE. Then, it is simple to show that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{F}$ have the following (biinfinite) matrix representations

$$\mathcal{A} = \begin{bmatrix}\ddots & & \\ & -n^2+1 & \\ & & \ddots\end{bmatrix}, \quad \mathcal{B} = \mathcal{C} = \begin{bmatrix}\ddots & & \\ & 1 & \\ & & \ddots\end{bmatrix},$$

$$\mathcal{F} = \frac{1}{2}\begin{bmatrix}\ddots & \ddots & & \\ \ddots & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots\end{bmatrix}.$$

[1]One can think of this problem as being the spatial equivalent of "vibrational control" in time-periodic systems. Here, we are interested in designing a spatially periodic feedback gain that will stabilize the system.

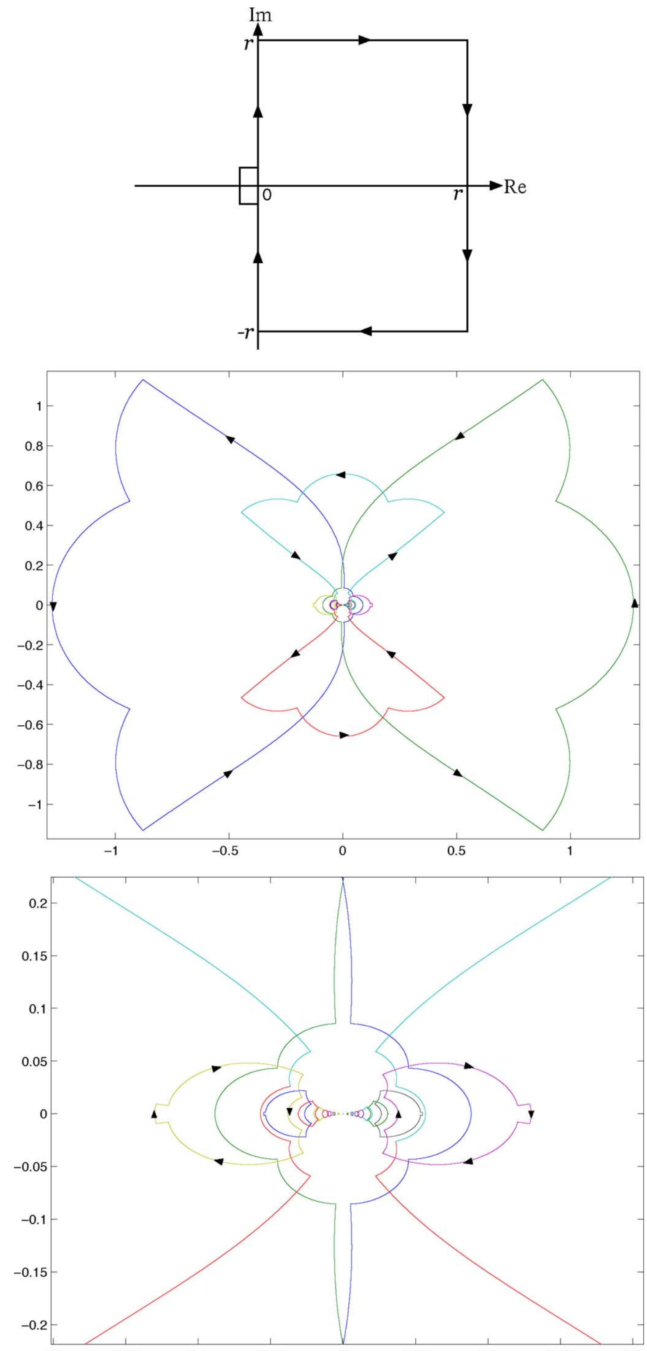

Fig. 3. Top: The Nyquist path $\mathfrak{D}$. Center: The eigenloci plot. Bottom: Blown-up version of the center part of the eigenloci plot.

Since $\Sigma(\mathcal{A}) = \{-n^2+1, n \in \mathbb{Z}\}$, and $\mathcal{A}$ is diagonal, then the Assumption (∗) is satisfied. For any $s \notin \Sigma(\mathcal{A})$, we have $(s\mathcal{I} - \mathcal{A})^{-1} = \mathrm{diag}\{\cdots, \frac{1}{s+n^2-1}, \cdots\}$. Thus $\sum_{n\in\mathbb{N}} \sigma_n\left((s\mathcal{I}-\mathcal{A})^{-1}\right) = \sum_{n\in\mathbb{Z}} |\frac{1}{s+n^2-1}| < \infty$. Hence, $(s\mathcal{I}-\mathcal{A})^{-1} \in B_1$ and Assumption (∗∗) is satisfied. We can now use the Nyquist stability criterion developed in Section III. We demonstrate that by plotting the eigenloci of $\widetilde{\mathcal{G}}(s)$ one can read off from this plot the stability of the closed-loop system for any value of $\gamma \in \mathbb{C}$.

$\lambda = 0, 0, 1$ are the eigenvalues of $\mathcal{A}$ inside $\mathfrak{D}$, where $\mathfrak{D}$ is the Nyquist path shown in Fig. 3 (top) that avoids the open-loop eigenvalues at 0. Since $p_+ = 3$, we need three counterclockwise encirclements of $-1/\gamma$ to achieve closed-loop stability. As can be seen in Fig. 3 (center) and its blown-up version Fig. 3 (bottom), no real value of γ yields a stable closed-loop system. In fact the only values of γ that yield a stable closed-loop system are those for which $-1/\gamma$ is in the neighborhood of the interval $-0.2j \leq -1/\gamma \leq 0.2j$ on the imaginary axis, where $-1/\gamma$ is encircled three times by the eigenloci.

V. Conclusion

We develop an extension of the argument principle and the Nyquist stability criterion that is applicable to distributed systems with possibly infinite-dimensional input and output spaces. The infinitesimal generator $\mathcal{A}$ of the system can be any unbounded operator that is sectorial with trace-class resolvent, and the input and output operators $\mathcal{B}$ and $\mathcal{C}$ are bounded. A direction for future research is to consider systems in which $\mathcal{B}$ and $\mathcal{C}$ are allowed to be unbounded operators.

Appendix

Proof of Theorem 1: From Assumption $(*)$ we know that $\mathcal{A}$ is sectorial. Then, because $\mathcal{B}, \mathcal{C}$, and $\mathcal{F}$ are bounded operators, it follows from [7, Thm 4.5.7] that $\mathcal{A}^{\mathrm{cl}} = \mathcal{A} - \gamma\mathcal{BFC}$ is sectorial for all $\gamma \in \mathbb{C}$.

Since $\mathcal{A}^{\mathrm{cl}}$ is sectorial, it defines an analytic C_0 semigroup and $e^{\mathcal{A}^{\mathrm{cl}}t}$ is differentiable for $t > 0$ [15], [16]. Then [17] shows that this is sufficient for the spectrum-determined growth condition to hold. Since $\Sigma(\mathcal{A}^{\mathrm{cl}}) \subset \mathbb{C}^-$ and $\Sigma(\mathcal{A}^{\mathrm{cl}})$ belongs to a left sector, it follows that $\Sigma(\mathcal{A}^{\mathrm{cl}})$ is bounded away from the imaginary axis. Let $\omega_\sigma = \sup_{z\in\Sigma(\mathcal{A}^{\mathrm{cl}})} \mathrm{Re}(z)$. Then, $\omega_\sigma < 0$ and $\mathcal{A}^{\mathrm{cl}}$ generates an exponential stable C_0 semigroup. Clearly, $\Sigma(\mathcal{A}^{\mathrm{cl}}) \subset \mathbb{C}^-$ is also a necessary condition for exponential stability of the closed-loop system, and the proof is complete. ■

To prove Theorem 2, we need the following two lemmas.

Lemma A1: For $s \in \rho(\mathcal{A})$, the determinant $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]$ is analytic in both γ and s.

Proof: For $s \in \rho(\mathcal{A})$, we have $\gamma\widetilde{\mathcal{G}}(s) \in B_1(\mathfrak{Y})$. Also $\gamma\widetilde{\mathcal{G}}(s) = \gamma\widetilde{\mathcal{C}}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}$ is clearly analytic in both γ and s for $s \in \rho(\mathcal{A})$. Then, it follows from [10, p. 163] that $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]$ too is analytic in both γ and s for $s \in \rho(\mathcal{A})$. ■

Lemma A2: The operators $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$ have discrete spectrum with no finite accumulation points.

Proof: The boundedness of the operators $\mathcal{B}, \mathcal{C}$, and $\mathcal{F}$ together with identity

$$(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1} - (s\mathcal{I} - \mathcal{A})^{-1} = -\gamma(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1}\mathcal{BFC}(s\mathcal{I} - \mathcal{A})^{-1} \quad \text{for all } s \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\mathrm{cl}})$$

implies that since $(s\mathcal{I} - \mathcal{A})^{-1} \in B_1(\mathfrak{X})$ by Assumption $(**)$, then $(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1} \in B_1(\mathfrak{X})$. From $(s\mathcal{I} - \mathcal{A})^{-1} \in B_1(\mathfrak{X})$, $(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1} \in B_1(\mathfrak{X})$, and $B_1(\mathfrak{X}) \subset B_\infty(\mathfrak{X})$, it follows that the spectrum of both $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$ consist entirely of isolated eigenvalues with no finite accumulation points [6, p. 187]. ■

Proof of Theorem 2: Consider any point s in $\mathfrak{D}$. Since $\mathfrak{D}$ does not pass through any eigenvalues of $\mathcal{A}$, then $s \in \rho(\mathcal{A})$, and thus, $\gamma\widetilde{\mathcal{C}}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B} \in B_1(\mathfrak{Y})$. Then, from [11], the operator $\left(\mathcal{I} + \gamma\widetilde{\mathcal{C}}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}\right)^{-1}$ exists and belongs to $B(\mathfrak{X})$ iff $\det[\mathcal{I} + \gamma\widetilde{\mathcal{C}}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}] \neq 0$, which is satisfied by the assumption. Applying an operator version of the matrix inversion lemma to $\left(\mathcal{I} + \gamma\widetilde{\mathcal{C}}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}\right)^{-1}$, we conclude that $(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1} = (s\mathcal{I} - \mathcal{A} + \gamma\mathcal{B}\widetilde{\mathcal{C}})^{-1} \in B(\mathfrak{X})$, and thus, $s \in \rho(\mathcal{A}^{\mathrm{cl}})$. Therefore, $\mathfrak{D}$ is contained inside $\rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\mathrm{cl}})$.

Let the path $\mathfrak{C}$ be that traversed by $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]$ as s travels once around $\mathfrak{D}$. By Lemma A1, the determinant $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]$ is analytic in s, and if $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)] \neq 0$ on $\mathfrak{D}$, we have

$$\begin{aligned} C\left(0; \det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]\big|_{s\in\mathfrak{D}}\right) &= \frac{1}{2\pi j}\int_{\mathfrak{C}} \frac{dz}{z} \\ &= \frac{1}{2\pi j}\int_{\mathfrak{D}} \frac{\frac{d}{ds}\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]}{\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]} ds \\ &= \frac{1}{2\pi j}\int_{\mathfrak{D}} \frac{d}{ds}\ln\Delta_{\mathcal{A}^{\mathrm{cl}}/\mathcal{A}}(s) ds \\ &= \frac{1}{2\pi j}\int_{\mathfrak{D}} \mathrm{tr}[(s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1} - (s\mathcal{I} - \mathcal{A})^{-1}] ds \end{aligned} \tag{A1}$$

where we have used (5) in the last equality.

By Lemma A2, the spectrum of the operators $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$ have no finite accumulation points. Therefore, the path $\mathfrak{D}$ encloses a finite number of the eigenvalues of $\mathcal{A}$ and $\mathcal{A}^{\mathrm{cl}}$. Thus, in the expression

$$\frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1} ds - \frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A})^{-1} ds$$

each term is a finite-dimensional projection [10, p. 11, p.15]. Taking the trace, from [6], it follows that

$$\mathrm{tr}\left[\frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A}^{\mathrm{cl}})^{-1} ds\right] - \mathrm{tr}\left[\frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A})^{-1} ds\right] \tag{A2}$$

is equal to the number of eigenvalues of $\mathcal{A}$ in $\mathfrak{D}$ minus the number of eigenvalues of $\mathcal{A}^{\mathrm{cl}}$ in $\mathfrak{D}$, where $\mathfrak{D}$ is the (clockwise) Nyquist path and is taken arbitrarily large to enclose $\mathbb{C}^+$. Finally (A1) and (A2) together give the required result. ■

Proof of Lemma 4: For $s \in \mathfrak{D} \subset \rho(\mathcal{A})$ the determinant $\det[\mathcal{I} + \gamma\widetilde{\mathcal{G}}(s)]$ is analytic in both γ and s by Lemma A1. The proof now proceeds exactly as in [4, p. 140] and is omitted. ■

Acknowledgment

The authors wish to thank the anonymous reviewers for many valuable suggestions; in particular, for comments which led to a revision of Theorem 1.

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Decentralized Control of Discrete-Event Systems When Supervisors Observe Particular Event Occurrences

Ying Huang, Karen Rudie, and Feng Lin

***Abstract*—Work on decentralized discrete-event control systems is extended to handle the case when, instead of always observing or never observing an event, a supervisor may observe only *some* occurrences of a particular event. Results include a necessary and sufficient condition for solving this version of the decentralized problem (which is analogous to the *co-observability* property used in the standard version of the problem) and a method for checking when this condition holds. In this paper, whether an event is observed by a given agent is dependent on that agent's state (or the string of events that agent has seen so far). This model of event observation is applicable to problems where a supervisor communicates observations of event occurrences to another supervisor to help the other one make control decisions.**

***Index Terms*—Decentralized control, discrete-event systems, partial observation, supervisory control.**

I. Introduction

A discrete-event system (DES) is a system that changes its state upon the occurrence of an event. The states in the system have symbolic values instead of numerical values, as in traditional continuous systems. In the early 1980s, Ramadge and Wonham initiated the framework of modeling and synthesis of controllers (supervisors) for *discrete-event systems* [5]. The standard formulation of DES control has been widely extended in a number of ways that include modular supervisory control, hierarchical supervisory control, timed DESs, dynamic DESs, and partial observation control. We deal with decentralized DES control problems [2], [9] in this paper. The standard work on decentralized DES assumes that a supervisor makes control decisions based only on its own, direct observations, i.e., it does not acquire any information about the plant to be controlled from other parties. If, however, co-observability is violated, exchanging information between supervisors may help to ensure that for every event that needs to be disabled, at least one supervisor has sufficient information about the plant to know to disable that event. In existing work on using communication to help supervisors make control decisions, one model of communication assumes that what is communicated is the observation of some event occurrence [6]. Since there are cases where it is desirable to minimize communication (for security or cost reasons), it is also desirable to have decentralized control solutions where only *some* occurrences of a given event are communicated. In this context, we can think of message reception by a supervisor as an (indirect) observation that some event has occurred.

Manuscript received January 26, 2006; revised August 13, 2007. Recommended by Associate Editor J. E. R. Cury. This work was supported in part by the Natural Sciences and Engineering Research Council (NSERC) and in part by the National Science Foundation (NSF) under Grant INT-0213651.

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Digital Object Identifier 10.1109/TAC.2007.914234

This paper examines the problem of finding decentralized supervisors to control a discrete-event plant, where the supervisors may observe particular occurrences of each event. We define a property called *state-based co-observability* which, together with controllability, is necessary and sufficient to solve the corresponding decentralized supervisory control problem. We also present a method for checking state-based co-observability. A version of this work first appeared in [3]. All proofs have been omitted due to page limits. However, they can be found in the full version of the paper, available at the Web site: http://ece.eng.wayne.edu/~flin

II. Background

A DES is an abstract process that is characterized by sequences of actions or events. When the system operates freely, without any interference or control, we call it a *plant*. A plant may generate some undesirable sequences, called *illegal behavior*. There are several formalisms used to model DESs. A commonly used model is the *finite-state machine* (FSM) (or *automaton*). In this model, a plant is modeled as a four-tuple $G = (\Sigma, Q^G, \delta^G, q_0^G)$, where Σ is the alphabet, Q^G is a finite set of states, $\delta^G\colon \Sigma \times Q^G \to Q^G$ is a partial function called a *transition function*, and q_0^G is the initial state of the plant. The transition function δ^G can be naturally extended to a partial function $\Sigma^* \times Q^G$, where Σ^* represents all possible finite strings over Σ, including ϵ. The *language generated by a plant* G, denoted by $L(G)$, is a language $L(G) := \{t \in \Sigma^* | \delta^G(t, q_0^G)!\}$, where $\delta^G(t, q_0^G)!$ stands for "$\delta^G(t, q_0^G)$ is defined." It represents all possible sequences that the plant G can generate.

If we denote concatenation of two strings s and t by st, then s is called a *prefix* of the string st. For a language $L \in \Sigma^*$, $\bar{L}$ is the set of all prefixes of strings in L. A language is *prefix-closed* if $L = \bar{L}$.

An uncontrolled plant may generated undesirable behaviors. We use an automaton, denoted by E, to represent the legal behavior of a plant G: $E = (\Sigma, Q^E, \delta^E, q_0^E)$. Without loss of generality, we assume that $L(E) \subseteq L(G)$ and that E is a subautomaton of G [1].

To force a plant G to behave in a legal way, we need a controller or supervisor, named S, to control some event occurrences based on its view of the plant's behavior. The supervisor may not have the ability to control all the events in the alphabet Σ; therefore, Σ is partitioned into two disjoint subsets Σ_c and Σ_{uc}, which comprise the *controllable events* and the *uncontrollable events*, respectively.

Formally, a supervisor S is a pair (T, ψ), where $T = (\Sigma, X, \xi, x_0)$ is an automaton and $\psi\colon \Sigma \times X \to \{0, 1\}$ is a feedback map. In ψ, the number 0 represents a disable control action and the number 1 represents an enable action. Since uncontrollable events cannot be disabled, we require that for all $\sigma \in \Sigma_{uc}$, for all $x \in X, \psi(\sigma, x) = 1$.

With the supervision of S, a plant G behaves in a constrained way, which is described by an automaton $S/G := (\Sigma, Q^G \times X, (\delta^G \times \xi)^\psi, (q_0^G, x_0))$, where $(\delta^G \times \xi)^\psi : \Sigma \times Q^G \times X \to Q^G \times X$ is defined as follows:

$$(\delta^G \times \xi)^\psi(\sigma, q, x) := \begin{cases} (\delta^G(\sigma, q), \xi(\sigma, x)) & \text{if } \delta^G(\sigma, q)!, \xi(\sigma, x))! \text{ and } \psi(\sigma, x) = 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$